Homework # 9 - [Due on December 3rd, 2018]

1. Consider a monopolist with cost function $C(q) = cq$, where $c > 0$ denotes its constant marginal production cost. The monopolist faces an inverse demand curve $p(q)$, which is strictly decreasing in output, $p'(q) < 0$, whose vertical intercept $p(0)$ satisfies $p(0) > c$.

   (a) Set up the monopolist profit-maximization problem and find its first-order condition. Interpret.

   - The monopolist solves the following profit-maximization problem,

     $$\max_{q \geq 0} \pi(q) = p(q)q - cq$$

     Differentiating with respect to q yields,

     $$p'(q)q + p(q) - c \leq 0$$

     Since $p(0) > c$ by definition, the problem has an interior solution, i.e., $q > 0$. Hence, the above expression can be rearranged in terms of the standard marginality condition

     $$p'(q)q + p(q) = c \tag{1}$$

     where the term on the left-hand side represents the marginal revenue from increasing output q, whereas the term on the right-hand side reflects marginal costs of increasing production. In addition, note that the marginal revenue, $MR \equiv p'(q)q + p(q)$ lies below the inverse demand function $p(q)$ since $p'(q) < 0$ by definition.

   (b) How is monopoly price affected by a marginal increase in marginal cost c when the inverse demand function is: (i) strictly concave; (ii) strictly convex; and (iii) linear?

   - *Roadmap of the proof:* We first analyze how an increase in the constant marginal cost c affects equilibrium output for each of the three cases mentioned above, and then describe how such variation in output affects the equilibirum price.
   - If marginal cost c increases marginally, the left-hand side of expression (1), marginal revenue, must also increase to guarantee that the equality in (1) holds. Differentiating the marginal revenue expression, we obtain

     $$p''(q)q + p'(q) + p'(q) = p''(q)q + 2p'(q) \tag{2}$$

which we now separately evaluate in three types of inverse demand functions: (i) strictly concave; (ii) strictly convex; and (iii) linear.

- *Case 1. Concave Demand Curve.* If the inverse demand function $p(q)$ is strictly concave, $p''(q) < 0$, implying that expression (2) becomes

$$\underbrace{p''(q)q}_{-} + \underbrace{2p'(q)}_{-} < 0$$

Therefore, expression (2) is unambigously decreasing, entailing that, when the marginal cost c increases, the monopolist responds decreasing its output q, ultimately leading to an increase in monopoly prices.

Parametric example. Consider inverse demand function $p(q) = a - bq^2$, , where $a, b > 0$, which is decreasing in output since $p'(q) = -2bq < 0$ and strictly concave given that $p''(q) = -2b < 0$. Therefore, evaluating expression (2) in this setting we find

$$(-2b)(q) + 2(-2bq) = -6 < 0$$

thus confirming our above result about marginal revenue being decreasing in output q when the inverse demand curve is strictly concave.

- *Case 2. Linear Demand Curve.* If the inverse demand function $p(q)$ is linear, $p''(q) = 0$, implying that expression (2) simplifies to

$$\underbrace{0}_{0} + \underbrace{2p'(q)}_{-} = \underbrace{2p'(q)}_{-} < 0$$

yielding the same results as in Case 1: Expression (2) is unambigously decreasing, entailing that, when the marginal cost c increases, the monopolist responds decreasing its output q, ultimately leading to an increase in monopoly prices.

Parametric example. Consider, for instance, the inverse demand function $p(q) = a - bq$, , where $a, b > 0$, which is linear in output since $p'(q) = -b$ and $p''(q) = 0$. We can then evaluate expression (2) in this setting to obtain

$$0q + 2(-b) = -2b < 0$$

thus confirming our above result about marginal revenue being decreasing in output q when the inverse demand curve is linear.

- *Case 3. Convex Demand Curve.* If the inverse demand function $p(q)$ is linear, $p''(q) > 0$, implying that expression (2) becomes

$$\underbrace{p''(q)q}_{+} + \underbrace{2p'(q)}_{-}$$

This result gives rise to two cases:

- (a) $|2p'(q)| > |p''(q)q|$, which implies that marginal revenue curve is still downward sloping; and
- (b) $|2p'(q)| < |p''(q)q|$, which entails that the marginal revenue curve is positively sloped.

The last case is, however, incompatible with the finding that the marginal revenue curve lies below the demand curve, and thus must have a negative slope and be steeper than the demand curve. Therefore, only case (a) can be sustained, which yields the same results as in Cases 1-2: the monopolist's marginal revenue curve is still downward sloping, implying that a marginal increase in c leads the monopolist to reduce its output level, ultimately increasing monopoly prices.

Parametric example. Consider, for instance, the inverse demand function $p(q) = a - b\ln q$, where $a, b > 0$, which is decreasing in output since $p'(q) = -\frac{b}{q} < 0$ and strictly convex given that $p''(q) = \frac{b}{q^2} > 0$. We can then evaluate expression (2) in this setting to obtain

$$\left(\frac{b}{q^2}\right)q + 2\left(-\frac{b}{q}\right) = -\frac{b}{q}$$

thus confirming our above result about marginal revenue being decreasing in output q when the inverse demand curve is convex.

2. Consider a monopolist facing inverse demand function $p(q) = 1 - q$; a supply function of $q = ax$, where x denotes the number of input that the monopolist hires (e.g., labor) and $a > 0$; and cost function $C(x) = bx + dx^2$, where $b, d > 0$, thus being increasing and convex in input units x.

   (a) Write down the monopolist's profit-maximization problem. Find the equilibrium values of the monopolist's input decision, and its output level.

   - The monopolist solves

$$\max_{x\geq 0} \pi = \underbrace{p(q)q}_{\text{Revenues}} - \underbrace{\left(bx + dx^2\right)}_{\text{Costs}} = (1 - ax)ax - \left(bx + dx^2\right)$$

Differentiating with respect to x, yields

$$-b - a^2x - 2dx + a(1 - ax) = 0$$

and solving for x, we obtain an equilibrium input level of

$$x^* = \frac{a - b}{a^2 + d}$$

Since $q = ax$ by definition, the equilibrium output that the monopolist produces is

$$q^* = ax^* = \frac{a(a - b)}{a^2 + d}.$$

(b) Assume now that the firm operates in a perfectly competitive industry, where price equals marginal cost. Find in this context the equilibrium values of the monopolist's input decision, and its output level.

- Setting $p(q) = MC(q)$ in this perfectly competitive market, we obtain

$$1 - q = b + 2dx,$$

and since $q = ax$ by definition, we can rewrite the above equality as

$$1 - ax = b + 2dx.$$

Solving for x, we find a perfectly-competitive input demand of

$$x^C = \frac{1 - b}{a + 2d}$$

Therefore, equilibrium output in this context becomes

$$q^C = ax^C = \frac{a(1 - b)}{a + 2d}.$$