EconS 501 Final Exam - December 10th, 2018

Show all your work clearly and make sure you justify all your answers.
NAME__________________________

1. Consider the market for smart pencil in which only one firm (SuperLapiz) enjoys a monopoly power. SuperLapiz faces two types of consumer: High and Low. The high-type consumer highly values to draw and sketch with full pressure sensitivity. However, the low-type consumer does not care too much about precision. The utility function for each type of consumer is

$$U_i(x_i, P_i) = v_i \left(x_i - \frac{2x_i^{\frac{1}{2}}}{v_i} \right) - F_i$$

where $i = \{H, L\}$, v_i is the consumer i's valuation for the good, where $v_H > v_L$, x_i is the quantity consumed and F_i is the price. SuperLapiz offers a menu of two-part tariffs to each type of consumer. Assume that β represents the proportion of high-type consumers, where $\beta \in [0, 1]$. In addition, SuperLapiz's marginal production cost is dentoted by c where $v_i > c$.

a. Set up SuperLapiz's profit-maximization problem, and show that the Participation Constraint for the high-type (PC_H) and the Incentive Compatibility for the low-type (IC_L) are binding.

$$\max \beta \left[F_H - cx_H \right] + (1 - \beta) \left[F_L - cx_L \right]$$

The participation constraints are

$$v_L \left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_L} \right) \geqslant F_L \tag{PC$_L$}$$

$$v_H \left(x_H - \frac{2x_H^{\frac{1}{2}}}{v_H} \right) \geqslant F_H \tag{PC$_H$}$$

and the incentive compatibility conditions are

$$v_L \left[\left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_L} \right) - \left(x_H - \frac{2x_H^{\frac{1}{2}}}{v_L} \right) \right] + F_H \geqslant F_L \tag{IC$_L$}$$

$$v_H \left[\left(x_H - \frac{2x_H^{\frac{1}{2}}}{v_H} \right) - \left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_H} \right) \right] + F_L \geqslant F_H \tag{IC$_H$}$$

We can show by contradiction that PC_H and IC_L are binding as proved during class.

1. b. Solve the monopolist's maximization problem with respect to x_H and x_L. Obtain the values of x_H and x_L and discuss under which conditions they are strictly positive.

$$\max \beta \left[F_H - cx_H\right] + (1-\beta)\left[F_L - cx_L\right]$$

since we consider that IC_H and PC_L bind, then:

$$F_L = v_L\left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_L}\right) \text{ and } F_H = v_H\left[\left(x_H - \frac{2x_H^{\frac{1}{2}}}{v_H}\right) - \left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_H}\right)\right] + v_L\left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_L}\right)$$

therefore the maximization problem becomes

$$\max_{x_L, x_H} \beta\left[v_H\left[\left(x_H - \frac{2x_H^{\frac{1}{2}}}{v_H}\right) - \left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_H}\right)\right] + v_L\left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_L}\right) - cx_H\right] + (1-\beta)\left[v_L\left(x_L - \frac{2x_L^{\frac{1}{2}}}{v_L}\right) - cx_L\right]$$

Taking first order conditions we obtain

$$\begin{aligned} x_L &: -\beta v_H + \beta x_L^{-\frac{1}{2}} + \beta v_L - \beta x_L^{-\frac{1}{2}} + (1-\beta)v_L - (1-\beta)x_L^{-\frac{1}{2}} - (1-\beta)c = 0 \\ x_H &: \beta v_H - \beta x_H^{-\frac{1}{2}} - \beta c = 0 \end{aligned}$$

therefore,

$$\begin{aligned} x_H &= (v_H - c)^{-2} \\ x_L &= \left[\frac{v_L}{(1-\beta)} - \frac{\beta v_H}{(1-\beta)} - c\right]^{-2} \end{aligned}$$

which are both positive amounts if $v_H > v_L > (1-\beta)c + \beta v_H$, where $v_i > c$ and $v_H > v_L$ by assumption.

1. c. Compare the output levels obtained in part (b) with the socially optimal output.

Socially optimal outputs can be found setting marginal utility equal to marginal cost, hence $q_i^{SO} = (v_i - c)^{-2}$, where $i = \{H, L\}$. Therefore in the case of the high-type consumer the output level coincides with the socially optimal output. However, the low-type consumes less units since

$$\begin{aligned} \left[\frac{v_L}{(1-\beta)} - \frac{\beta v_H}{(1-\beta)} - c\right]^{-2} &< (v_L - c)^{-2} \\ \frac{v_L}{(1-\beta)} - \frac{\beta v_H}{(1-\beta)} - c &< (v_L - c) \\ v_L - (1-\beta)v_L &< \beta v_H \\ \beta v_L &< \beta v_H \end{aligned}$$

which holds since $\beta \in [0, 1]$.

2 A utility function is additively separable if it has the form

$$u(x) = \sum_{k=1}^{L} u_k(x_k)$$

For instance, in the context of three goods, an additively separable function would be $u(x) = u_1(x_1) + u_2(x_2) + u_3(x_3)$, where function $u_k(x_k)$ can be linear or nonlinear in the units of good k, x_k.

(a) For the context of two goods, provide at least two examples of utility functions that are additively separable, and two examples of utility functions that are not.

- First, note that the utility of good k, $u_k(x_k)$, could be $u_k(x_k) = ax_k$, $u_k(x_k) = a \ln x_k$ or $u_k(x_k) = ax_k^2$ where $a > 0$; or more generally, functions of the form $u_k(x_k) = ax_k^{\beta}$, where $a, \beta \in \mathbb{R}$.
- *Additively separable utility functions.* Consider, for instance, goods that are regarded as subsitutes, with utility function

$$u(x, y) = ax + by,$$

where $a, b \in \mathbb{R}$; or, more generally, utility functions such as

$$u(x, y) = ax^{\beta} + by^{\delta},$$

where $a, b, \beta, \delta \in \mathbb{R}$. Note that the last example includes quasi-linear utility functions of the form

$$u(x, y) = ax^{\beta} + by,$$

as a special case (when $\delta = 1$).
- *Not additively separable utility functions.* Consider, for example, the Cobb-Douglas utility function

$$u(x, y) = ax^{\alpha}y^{\beta},$$

where $a, b, \alpha, \beta \in \mathbb{R}$; the utility function representing goods that are regarded as complements,

$$u(x, y) = \min\{ax, by\},$$

where $a, b \in \mathbb{R}$; or the Stone-Geary utility function

$$u(x, y) = a\left(x - \overline{x}\right)^{\alpha}\left(y - \overline{y}\right)^{\beta},$$

where $a, \alpha, \beta \in \mathbb{R}$ and $\overline{x}, \overline{y} > 0$.

(b) Show that the marginal utility of good k is a function of the units of good k alone. Interpret.

- Differentiating the utility function $u(x)$ with respect to x_k, we obtain

$$MU_k = \frac{\sum_{l \neq k} u_l(x_l)}{\partial x_k} = \frac{\partial u_k(x_k)}{\partial x_k}$$

since all other components of the utility function, $\sum_{l \neq k} u_l(x_l)$, do not include x_k as arguments. Intuitively, when increasing the units of good k, the consumer only cares about the additional utility he obtains from this good, but ignores the number of units of other goods he consumes (that is, there is no interaction between the utilities of different goods, nor on their marginal utilities). Denoting the marginal utility of good k as $u_k'(x_k(p,w))$, we can more compactly express our above result as

$$MU_k = u_k'(x_k(p,w))$$

(c) Show that the Walrasian and Hicksian demand functions imply that all goods must be normal rather than inferior. (For simplicity, you can assume that the utility of every good is strictly concave for every good, differentiability, and interior solutions.)

- First, we know that the following tangency condition holds both in the UMP and in the EMP

$$MRS_{k,l} = \frac{MU_k}{MU_l} = \frac{u_k'(x_k(p,w))}{u_l'(x_l(p,w))} = \frac{p_k}{p_l}$$

(Recall that both the UMP and EMP have this tangency condition between the indifference curve and the budget line in common. However, the UMP inserts this result into its contraint, the budget line; whereas the EMP inserts the above result into its constraint, the utility level that the individual must reach.) Rearranging the above tangency condition, we obtain

$$u_k'(x_k(p,w)) = \frac{p_k}{p_l} u_l'(x_l(p,w)).$$

- Since we seek to show that no good can be inferior, we must consider a wealth change, leaving all prices unchanged. If wealth w increases, the demand for at least one good (say, good l) has to increase by Walras' law (otherwise, the individual would be buying fewer units of all goods, thus not exhausting his wealth). We seek to show that the demand for the remaining good k must also increase, thus implying that all goods are normal.
- To see this, first note that if the demand for good l increases, its marginal utility decreases. Graphically, a decrease in $u_l'(x_l(p,w))$ implies that the line representing $\frac{p_k}{p_l} u_l'(x_l(p,w))$ shifts downwards, yielding a new crossing point to the right-hand side of the initial crossing point depicted in the above figure. As a consequence, the consumer demands a larger amount of good k, i.e., $x_k(p,w)$ increases, ultimately implying that good k must be normal. Since our analysis applies to any good k, all goods must be normal.

3. Consider a setting with coupon rationing so that each commodity has two prices: a dollar price and a ration-coupon price. Assume that there are three commodities and that the consumer has a dollar income w and a ration-coupon allotment z. Also assume that this allotment is not so liberal that any commodity combination that he can afford to purchase with his dollar income can also be purchased with his coupons.

(a) Set up this consumer's utility-maximization problem assuming a strictly concave utility function. Interpret your results. [*Hint*: You need to impose two constraints: a budget and a coupon constraint].

- The consumer solves

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & u(x_1,x_2,x_3) \\ \text{s.t.} \quad & p_1x_1 + p_2x_2 + p_3x_3 \le w \quad \text{(Budget constraint)} \\ & c_1x_1 + c_2x_2 + c_3x_3 \le z \quad \text{(Coupon constraint)} \end{aligned}$$

where $p_1 - p_3$ denote good prices, while $c_1 - c_3$ represent ration-coupon prices for each good. The associated Lagrangian is

$$L = u(x_1,x_2,x_3) + \lambda(w - p_1x_1 - p_2x_2 - p_3x_3) + \mu(z - c_1x_1 - c_2x_2 - c_3x_3)$$

where λ (μ) is the Lagrange multiplier for the budget (coupon) constraint.

(b) Find first-order conditions and interpret them.

- Differentiating with respect to x_1, x_2, x_3, λ and μ, we find

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= u_1 - \lambda p_1 - \mu c_1 \le 0 & x_1 \ge 0 \\ \frac{\partial L}{\partial x_2} &= u_2 - \lambda p_2 - \mu c_2 \le 0 & x_2 \ge 0 \\ \frac{\partial L}{\partial x_3} &= u_3 - \lambda p_3 - \mu c_3 \le 0 & x_3 \ge 0 \\ \frac{\partial L}{\partial \lambda} &= w - p_1x_1 - p_2x_2 - p_3x_3 \ge 0 & \lambda \ge 0 \\ \frac{\partial L}{\partial \mu} &= z - c_1x_1 - c_2x_2 - c_3x_3 \ge 0 & \mu \ge 0 \end{aligned}$$

For compactness, $u_i \equiv \frac{\partial u}{\partial x_i}$ denotes the marginal utility from good i. In the case of interior solutions, the first-order conditions hold with equality and yield

$$\underbrace{\frac{u_i}{u_j}}_{\text{MRS}} = \underbrace{\frac{\lambda p_i + \mu c_i}{\lambda p_j + \mu c_j}}_{\text{"Generalized" price ratio}}$$

where $j \ne i$. Intuitively, the left-hand side represents the ratio of marginal utilities between goods i and j, that is, marginal rate of substitution; as in standard consumer problems without coupons or rationing. The right-hand side, however, is not necessarily equal to the standard price ratio $\frac{p_i}{p_j}$, since it also includes good prices and ration-coupon prices, weighted by the corresponding marginal utilities (the Lagrange multipliers λ and μ). For compactness, this price ratio is often referred to as "Generalized price ratio" since it embodies the standard price ratio $\frac{p_i}{p_j}$ as a special case, when $\mu = 0$.

(c) Find a sufficient condition guaranteeing that the imposition of rationing does not alter the consumer's optimal bundle.

- As suggested in part (b) of the exercise, the imposition of rationing yields the standard optimality condition $MRS = \frac{p_i}{p_j}$ when $\mu = 0$. Intuitively, this occurs when the marginal utility of additional coupons is close to zero, which could happen when the consumer receives a large amount of coupons z. For instance, if z is weakly larger than the total amount of goods $x_1 + x_2 + x_3$ the consumer purchases under no rationing, then rationing has no effect on his purchases, making the standard optimality condition $MRS = \frac{p_i}{p_j}$ still valid.

(d) *Numerical Example*: Consider now that the consumer exhibits Cobb-Douglas utility function $u(x_1, x_2, x_3) = x_1 x_2 x_3$, income $w = 100$, and price vector $p = (1, 4, 2)$. Assume that, under rationing, the price vector is $c = (1/2, 3, 1)$ and the total coupon amount is $z = 80$. Find the optimal bundle with and without rationing in this context.

- *No rationing.* When no rationing is in effect, the consumer solves a standard utility maximization problem

$$\max_{x_1, x_2, x_3} \quad x_1 x_2 x_3$$
$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 + p_3 x_3 \leq w$$

with associated Lagrangian

$$L = x_1 x_2 x_3 + \lambda(w - p_1 x_1 - p_2 x_2 - p_3 x_3)$$

Taking first-order conditions, we obtain

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= x_2 x_3 - \lambda p_1 \leq 0, \text{ with equality if } x_1^* > 0 \\
\frac{\partial L}{\partial x_2} &= x_1 x_3 - \lambda p_2 \leq 0, \text{ with equality if } x_2^* > 0 \\
\frac{\partial L}{\partial x_3} &= x_1 x_2 - \lambda p_3 \leq 0, \text{ with equality if } x_3^* > 0 \\
\frac{\partial L}{\partial \lambda} &= w - p_1 x_1^* - p_2 x_2^* - p_3 x_3^* = 0
\end{aligned}$$

In the case of interior solutions, solving for λ yields the following relations

$$\begin{aligned}
\frac{x_2}{x_1} &= \frac{p_1}{p_2} \iff \frac{p_2 x_2}{p_1} = x_1 \\
\frac{x_3}{x_2} &= \frac{p_2}{p_3} \iff x_3 = \frac{p_2 x_2}{p_3} \\
\frac{x_2 x_3}{x_1 x_2} &= \frac{p_1}{p_3}
\end{aligned}$$

Substituting the above conditions into the budget constraint gives

$$\begin{aligned}
p_1 x_1 + p_2 x_2 + p_3 x_3 &= \\
p_1 \frac{p_2 x_2}{p_1} + p_2 x_2 + p_3 \frac{p_2 x_2}{p_3} &= w
\end{aligned}$$

Finally, solving for x_2 yields the Walrasian demand for good x_2, $x_2(w, p_1, p_2, p_3) = \frac{w}{3p_2}$. Similar manipulations gives the Walrasian demands for goods x_1 and x_3, $x_1(w, p_1, p_2, p_3) = \frac{w}{3p_1}$ and $x_3(w, p_1, p_2, p_3) = \frac{w}{p_3}$. Substituting income $w = 100$ and price vector $p = (1, 4, 2)$ entails Walrasian demands

$$\begin{aligned} x_1^* &= \frac{100}{3} \\ x_2^* &= \frac{100}{12} \\ x_3^* &= \frac{100}{6} \end{aligned}$$

- *Rationing.* When rationing is in effect, the consumer solves

$$\begin{aligned} &\max_{x_1,x_2,x_3} \quad x_1x_2x_3 \\ \text{s.t.} \quad & p_1x_1 + p_2x_2 + p_3x_3 \leq w \quad \text{(Budget constraint)} \\ & c_1x_1 + c_2x_2 + c_3x_3 \leq z \quad \text{(Coupon constraint)} \end{aligned}$$

with associated Lagrangian

$$L = x_1x_2x_3 + \lambda(w - p_1x_1 - p_2x_2 - p_3x_3) + \mu(z - c_1x_1 - c_2x_2 - c_3x_3)$$

Taking first-order conditions, we obtain

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= x_2x_3 - \lambda p_1 - \mu c_1 \leq 0, \text{ with equality if } x_1^* > 0 \\ \frac{\partial L}{\partial x_2} &= x_1x_3 - \lambda p_2 - \mu c_2 \leq 0, \text{ with equality if } x_2^* > 0 \\ \frac{\partial L}{\partial x_3} &= x_1x_2 - \lambda p_3 - \mu c_3 \leq 0, \text{ with equality if } x_3^* > 0 \\ \frac{\partial L}{\partial \lambda} &= w - p_1x_1 - p_2x_2 - p_3x_3 \geq 0 \quad \lambda \geq 0 \\ \frac{\partial L}{\partial \mu} &= z - c_1x_1 - c_2x_2 - c_3x_3 \geq 0 \quad \mu \geq 0 \end{aligned}$$

We then require

$$\begin{aligned} \lambda(w - p_1x_1 - p_2x_2 - p_3x_3) &= 0 \\ \mu(z - c_1x_1 - c_2x_2 - c_3x_3) &= 0 \end{aligned}$$

We solve this problem considering several cases:

- **Case 1**, $\lambda = 0, \mu > 0$. If the coupon constraint binds but the budget constrain does not, the first-order conditions become

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= x_2x_3 - \mu c_1 = 0 \\ \frac{\partial L}{\partial x_2} &= x_1x_3 - \mu c_2 = 0 \\ \frac{\partial L}{\partial x_3} &= x_1x_2 - \mu c_3 = 0 \\ \frac{\partial L}{\partial \mu} &= z - c_1x_1 - c_2x_2 - c_3x_3 = 0 \end{aligned}$$

Solving for μ yields the following relations

$$\begin{aligned}\frac{x_2}{x_1} &= \frac{c_1}{c_2} \Longleftrightarrow \frac{c_2 x_2}{c_1} = x_1 \\ \frac{x_3}{x_2} &= \frac{c_2}{c_3} \Longleftrightarrow x_3 = \frac{c_2 x_2}{c_3} \\ \frac{x_3}{x_1} &= \frac{c_1}{c_3}\end{aligned}$$

Substituting the above conditions into the coupon constraint gives

$$\begin{aligned}c_1 x_1 + c_2 x_2 + c_3 x_3 &= \\ c_1 \frac{c_2 x_2}{c_1} + c_2 x_2 + c_3 \frac{c_2 x_2}{c_3} &= z\end{aligned}$$

which yields the following Walrasian demands:

$$\begin{aligned}x_1^* &= \frac{z}{3c_1} = \frac{160}{3} \\ x_2^* &= \frac{z}{3c_2} = \frac{80}{9} \\ x_3^* &= \frac{z}{3c_3} = \frac{80}{3}\end{aligned}$$

However, if we substitute these demands into the budget constraint, we find that they violates the budget constraint since

$$\frac{160}{3} + 4\frac{80}{9} + 2\frac{80}{3} = 142 > 100$$

implying that the above Walrasian demands cannot be an optimal solution.

- **Case 2**, $\lambda > 0, \mu = 0$. In this case, the coupon constraint does not bind while the budget constrain binds. The first-order conditions then become

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= x_2 x_3 - \lambda p_1 = 0 \\ \frac{\partial L}{\partial x_2} &= x_1 x_3 - \lambda p_2 = 0 \\ \frac{\partial L}{\partial x_3} &= x_1 x_2 - \lambda p_3 = 0 \\ \frac{\partial L}{\partial \lambda} &= w - p_1 x_1 - p_2 x_2 - p_3 x_3 = 0\end{aligned}$$

which yields the same list of FOCs as without rationing. Therefore, Walrasian demands coincide with those under no rationing:

$$\begin{aligned}x_1^* &= \frac{100}{3} \\ x_2^* &= \frac{100}{12} \\ x_3^* &= \frac{100}{6}\end{aligned}$$

Substituting this solution into the coupon constraint, we find that the coupon constraint is met, that is, the consumer does not exhaust his coupons, since

$$\frac{1}{2}\frac{100}{3} + 3\frac{100}{12} + \frac{100}{6} = 58.33 < 80$$

4. Consider a setting with N individuals, where every individual i simultaneously and independently chooses his exploitation level $e_i \geq 0$. The marginal cost of effort is symmetric across individuals, $c > 0$. For compactness, denote by $e_{-i} = (e_1, ..., e_{i-1}, e_{i+1}, ...e_N)$ the profile of effort levels by i's rivals, E the sum of all individuals' efforts, and $E_{-i} = E - e_i$ the aggregate effort of all i's rivals. The utility function for individual i is given by

$$u_i(e_i, e_{-i}) = A(e_i + E_{-i})e_i - ce_i$$

In addition, assume that $A(\cdot)$ represents an outcome function which is strictly decreasing in aggregate effort E. This outcome function can represent several economic contexts, such as: (1) *a common-pool resource*, where $A(E) = \frac{f(E)}{E}$ indicates the average appropriation accruing to every player i, with $f(E)$ capturing total appropriation (e.g., total catches by all fishermen), as in Dasgupta and Heal (1979); (2) *Cournot competition*, where $A(E)$ represents the inverse demand function, which decreases in aggregate output, e.g., $A(E) = a - bE$; and (3) *rent-seeking contests* where $A(E) = \frac{e_i}{E}$ indicates the probability that player i wins the prize (e.g., promotion in a company), which is also decreasing in total effort.

(a) *Competitive equilibrium.* Find the implicit function that defines the equilibrium effort level that every individual i chooses. Define under which conditions the equilibrium effort you found is a utility maximum rather than a minimum.

- Every individual i solves

$$\max_{e_i} \quad A(e_i + E_{-i})e_i - ce_i$$

Taking first-order condition with respect to effort e_i, we obtain

$$A(E) + e_i A'(E) = c$$

In words, every individual i chooses an effort level e_i such that the marginal benefit from *individual* effort (left-hand side of the above equation) coincides with its own marginal cost from effort, c (right-hand side).

- Taking the second-order condition with respect to e_i, we obtain

$$2A'(E) + e_i A''(E)$$

Therefore, for the equilibrium effort to be a maximum, this second-order condition must be negative, i.e., $2A'(E) + e_i A''(E) < 0$. By definition, we know that the outcome of aggregate effort is decreasing, $A'(E) < 0$. If we assume that it is concave, $A''(E) < 0$, the second-order condition holds. Alternatively, function $A(E)$ can be convex as long as it is not "extremely convex", that is, $2A'(E) < e_i A''(E)$.

(b) *Social optimum.* Assume that a social planner considers the sum of every individual's utility as a measure of social welfare. Find the profile of effort levels that maximize social welfare (again, an implicit equation).

- The social planner's problem can be stated as

$$\max_{e_1,e_2,\dots e_N} \sum_{i=1}^{N} u_i(e_i, e_{-i}) = \sum_{i=1}^{N} A(e_i + E_{-i})e_i - ce_i$$

Taking the first-order conditions with respect to every e_i, we obtain

$$A(E) + A'(E)(e_1 + e_2 + ... + e_N) - c = 0$$

where $E = \sum_{i=1}^{N} e_i$ denotes aggregate effort. Rearranging this expression, we obtain the profile of effort levels that maximize social welfare, which is the solution to the following implicit equation

$$A(E) + EA'(E) = c$$

- Intuitively, this equations says that the social planner chooses the profile of effort levels $e^* = (e_1, e_2, ...e_N)$ such that the marginal benefit of the *aggregate* effort (left-hand side of the equation) coincides with the marginal cost, c (right-hand side).

(c) *Comparison.* Compare your results from parts (a) and (b), showing that equilibrium effort is socially excessive. Interpret your results in terms of the above three economic contexts discussed above.

- In part (a), considering the welfare of each player, the equilibrium effort is determined by the equation

$$A(E) + e_iA'(E) - c = 0 \tag{1}$$

In part (b), considering the sum of every individual's utility, the equilibrium effort is determined by the equation

$$A(E) + EA'(E) - c = 0 \tag{2}$$

Comparing equations (1) and (2), since $A(\cdot)$ is strictly decreasing in E, we can see that players' effort in the competitive equilibrium is socially excessive. As a result, the aggregate socially optimal effort, E^{SO}, is lower than the competitive equilibrium effort, E^*, implying that, $A(E^{SO}) > A(E^*)$. The high quantity of aggregate effort generated by the competitive equilibrium results in a welfare loss due to the presence of negative externalities.

- *Common-pool resource interpretation.* When outcome function $A(E) = \frac{f(E)}{E}$ represents the average appropriation accruing to every player i (e.g., tons of fish captured by fisherman i), the payoff function for each individual i can be written as

$$u_i(e_i, e_{-i}) = \frac{e_i}{E}f(E) - ce_i \tag{3}$$

- *Rent-seeking interpretation.* Alternatively, equation (3) can be interpreted as the payoff function in a rent-seeking contest with $\frac{e_i}{E}$ as the probability that player i wins the prize. Intuitively, individual i's probability of winning increases in his own effort e_i, but decreases in the effort other players exert, as captured by aggregate effort E. A higher aggregate effort implies a lower probability of winning the prize. Hence, given that $E^* > E^{SO}$, the probability of winning the contest is higher when maximizing the social welfare than when maximizing individual welfare.
- *Cournot competition interpretation.* When outcome function $A(E)$ represents the inverse demand function, the inefficiency of the noncooperative equilibrium relative to the social optimum is equivalent to showing that collusion leads to lower aggregate production ($E^{SO} < E^*$) but higher prices ($A(E^{SO}) > A(E^*)$). Hence, collusive equilibrium results in higher profits.