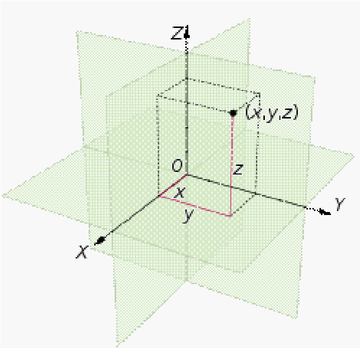
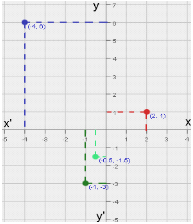
Euclidean Spaces

Chapter 10 -S&B



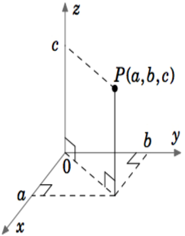
- The Real Line: every real number is represented by exactly one point on the line.

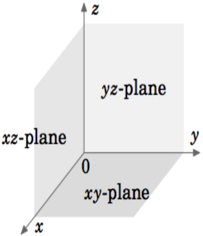
- The plane (i.e., consumption bundles): Pairs of numbers have a geometric representation *Cartesian plane* or *Euclidean 2-space*, $\mathbb{R}^2$.



Three Dimensions and More

- 3 dimensional Euclidean space, $\mathbb{R}^3$. Coordinate axis: the x-axis, the y-axis and the z-axis

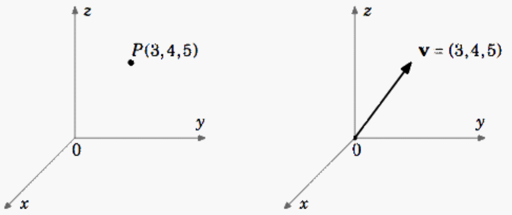




- $\mathbb{R}^1$ consists of single numbers; $\mathbb{R}^2$consists of ordered pairs of numbers and $\mathbb{R}^n$ the euclidean n-space, refers to how many numbers are needed to describe each location ($\mathbb{R}^5$ is (a, b, c, d, e))

Vectors

- We can think of n-tuples of number as locations (i.e. location in commodity space). We can also interpret n-tuples as displacements.
- We picture these displacements as arrows in $\mathbb{R}^n$. $(2, 3)$ means: move 3 units to the right and 2 units up from your current location.



(a) The point (3,4,5) (b) The vector (3,4,5)

- How do we assign an n-tuple to a particular arrow?

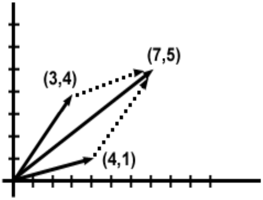
Vectors

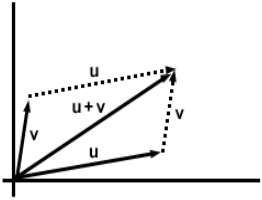
- If a displacement goes from initial location (a, b) to the terminal location (c, d).
- The move in the x_1-direction is $c - a$, since $a + (c - a) = a$.
- The move in the x_2-direction is $d - b$, since $b + (d - b) = d$.
- The displacement is $(c - a, d - b)$
- Generally, the displacement from the point $p(a_1, a_2, ..., a_n)$ to the point $q(b_1, b_2, ..., b_n)$ in $\mathbb{R}^n$ is written

$$\overrightarrow{pq} = (b_1 - a_1, b_2 - a_2, ..., b_n - a_n)$$

The Algebra of Vectors

- Addition and subtraction: $(3, 2) + (4, 1) = (7, 3)$
- $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
- We can only add two vectors from the same vector space; vector addition is commutative $u + v = v + u$





The Algebra of Vectors

- *Vector addition obeys the other rules which the addition of real numbers obeys:*
  - associative rule, the existence of a zero and the existence of an additive inverse.
- Zero vector represents no displacement at all;

$$0 = (0, 0, ..., 0)$$

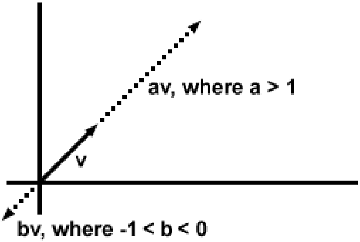
- Geometrically, it is a displacement $\overrightarrow{PP}$ having the same terminal point as initial point.
- If $u = (a_1, a_2, ..., a_n)$ the negative u is $(-a_1, -a_2, ..., -a_n)$; geometrically, one interchanges the head and the tail of u to obtain the head and tail of $-u$ $(-\overrightarrow{PQ} = \overrightarrow{QP})$

Scalar Multiplication

- *Go twice as far* or *you are halfway there*
- We multiply a vector by a real number (or scalar)

$$r \times x = (rx_1, ..., rx_n)$$

$$2 \times (1,1) = (2,2) \text{ or } \frac{1}{2} \times (-4,2) = (-2,1)$$



Scalar Multiplication

- Distributive laws in Euclidean spaces

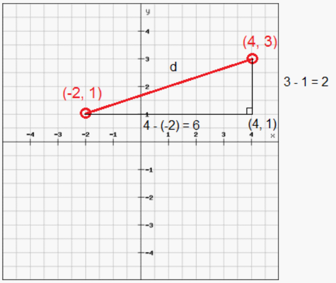
$$
\begin{aligned}
(r+s)u &= ru + su \\
r(u+v) &= ru + rv
\end{aligned}
$$

- where r and s are scalars and u and v are vectors.

Length and Inner Product in Rn

- The most basic geometric property is distance or length.
- Notation: the length of line segment $\overline{PQ}$ is denoted by the symbol $\|\overline{PQ}\|$
- Consider P and Q lie in the plane $\mathbb{R}^2$ and have the same x_2-coordinate. P has (a_1, b) and Q has (a_2, b)
- $\|\overline{PQ}\| = |a_2 - a_1|$

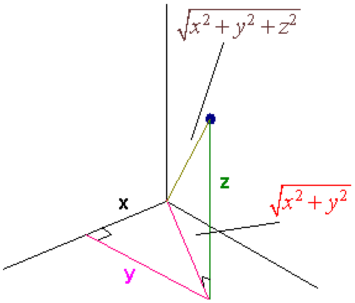
Length

- 

Length

- To find the distance from $P(a_1, b_1, c_1)$ to $Q(a_2, b_2, c_2)$ in $\mathbb{R}^3$, we use

$$\|PQ\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}$$

- 

Generalization

- If $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ are the coordinates of x and y, respectively, in Euclidean n-space, then

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$$

- If we take y to be 0, then

$$\|x\| = \sqrt{(x_1)^2 + (x_2)^2 + ... + (x_n)^2}$$

Scalar multiplication on the length

Theorem

$\|r\mathbf{v}\| = |r| \cdot \|\mathbf{v}\|$ *for all* r *in* $\mathbb{R}^1$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$

Proof.

$$\begin{aligned}\|r(v_1, ..., v_n)\| &= \|(rv_1, ..., rv_n)\| \\ &= \sqrt{(rv_1)^2 + ... + (rv_n)^2} \\ &= \sqrt{r^2(v_1^2 + ... + v_n{}^2)} \\ &= |r|\sqrt{(v_1^2 + ... + v_n{}^2)}, \text{ since } \sqrt{r^2} = |r| \end{aligned}$$

□

Unit Vector

- Given a non-zero displacement vector **v**, we will occasionally need to find a vector **w** which points in the same direction as **v**, but has length 1. Such a vector **w** is called the **unit vector**.
- To achieve such a vector **w**, premultiply **v** by the scalar $r = \frac{1}{\|\mathbf{v}\|}$

Example

For example the length of $(1, -2, 3)$ in $\mathbb{R}^3$ is

$$\begin{aligned} \|(1,-2,3)\| &= \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} \\ \frac{1}{\sqrt{14}}(1,-2,3) &= (\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}) \end{aligned}$$

It is a vector which points in the same direction as $(1, -2, 3)$ but has length 1.

The Inner Product (dot product or scalar product)

Definition

Let $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n)$ be two vectors in $\mathbb{R}^n$. The Euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$, written $\mathbf{u} \cdot \mathbf{v}$, is the number

$$\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + ... + u_n \cdot v_n$$

Example

if $\mathbf{u} = (4, -1, 2)$ and $\mathbf{v} = (6, 3, -4)$ then

$$\mathbf{u} \cdot \mathbf{v} = 4 \cdot 6 + -1 \cdot 3 + 2 \cdot -4 = 13$$

The Inner Product

Theorem

Let $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *be arbitrary vectors in* $\mathbb{R}^n$ *and let* r *be an arbitrary scalar. Then*

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u} \\
\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w} \\
\mathbf{u}\cdot(r\mathbf{v}) &= r(\mathbf{u}\cdot\mathbf{v}) = (r\mathbf{u})\cdot\mathbf{v} \\
\mathbf{u}\cdot\mathbf{u} &\geq \mathbf{0} \\
\mathbf{u}\cdot\mathbf{u} &= \mathbf{0} \quad \textit{implies } \mathbf{u}=\mathbf{0} \\
(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u}+2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v}
\end{aligned}
$$

The Inner Product

- The Euclidean inner product is closely connected to the Euclidean length of a vector

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \ldots + u_n^2 \text{ and } \|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$
$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

- Hence, the distance between two vectors **u** and **v** can be rewritten in terms of the inner product

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v})\,(\mathbf{u} - \mathbf{v})}$$

The Inner Product

Theorem

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors in the* $\mathbb{R}^n$*. Let* θ *the the angle between them. Then,*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$$

The Inner Product

Remarks: When the angle is a right angle, we say **u** and **v** are orthogonal

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n = 0$$

Triangle Inequality. It states that any side of a triangle is shorter than the sum of the lengths of the other two sides.

Theorem

For any two vectors **u** *and* **v** *in* $\mathbb{R}^n$*,*

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

The Inner Product

Theorem

For any two vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$*,*

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof.

Recall that $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\theta \leq 1$ then

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &\leq \|\mathbf{u}\|\,\|\mathbf{v}\| \\
\|\mathbf{u}\|^2 + 2\left(\mathbf{u}\cdot\mathbf{v}\right) + \|\mathbf{v}\|^2 &\leq \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} &\leq \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2 \\
\left(\mathbf{u}+\mathbf{v}\right)\left(\mathbf{u}+\mathbf{v}\right) &\leq \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2 \\
\|\mathbf{u}+\mathbf{v}\|^2 &\leq \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2 \\
\|\mathbf{u}+\mathbf{v}\| &\leq \|\mathbf{u}\| + \|\mathbf{v}\|
\end{aligned}$$

Line

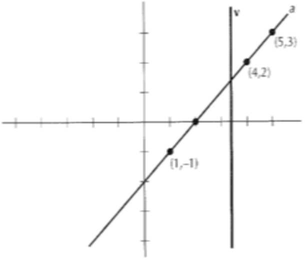
We will show how to describe lines and planes and their higher dimensional analogues. Let us work in $\mathbb{R}^2$

$$y = mx + b$$

- m is the slope
- b is the y-intercept

Line

What is the equation of line **v** in the figure?



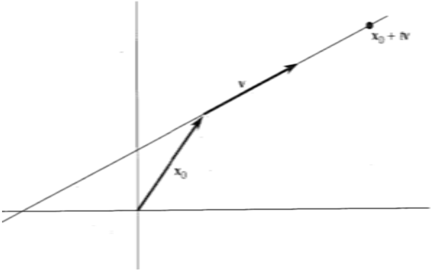
Lines in $\mathbf{R}^2$.

- We cannot solve for y in terms of x

Parametric Representation

- A parametric representation of a point on a line uses parameter t in the coordinate expression of the point
- Expression $(x_1(t), x_2(t))$ for some value t^* of the parameter t
- Think of t as representing time and the parametrization as describing the transversal of a path.
- $(x_1(t), x_2(t))$ describes the particular location which is reached at time t.

Parametric Representation



- A line is determined by: a point $\mathbf{x}_0$ on the line and a direction $\mathbf{v}$ in which to move from $\mathbf{x}_0$.
- Geometrically, to describe motion in the direction $\mathbf{v}$ we add scalar multiples of $\mathbf{v}$ to $\mathbf{x}_0$.

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$$

Parametric Representation

- Parameterization works in all dimensions. For example, the line in $\mathbb{R}^3$ through the point $\mathbf{x}_0 = (2, 1, 3)$ in the direction $\mathbf{v} = (4, -2, 5)$ has the parameterization

$$\begin{aligned} \mathbf{x}(t) &= (x_1(t), x_2(t), x_3(t)) \\ &= (2, 1, 3) + t(4, -2, 5) \\ &= (2 + 4t, 1 - 2t, 3 + 5t) \end{aligned}$$

Parametric Representation

- Another way to determine a line is to identify two points on the line.
- Suppose $\mathbf{x}$ and $\mathbf{y}$ lie on a line l.
- l can be viewed as the line which goes through $\mathbf{x}$ and points in the direction $\mathbf{y} - \mathbf{x}$.
- A parameterization for the line is

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \\ &= \mathbf{x} + t\mathbf{y} - t\mathbf{x} \\ &= (1 - t)\mathbf{x} + t\mathbf{y} \end{aligned}$$

- We parameterize the line segment joining $\mathbf{x}$ to $\mathbf{y}$ as

$$l(\mathbf{x}, \mathbf{y}) = \{(1 - t)\mathbf{x} + t\mathbf{y} : 0 \leq t \leq 1\}$$

Parametric Representation

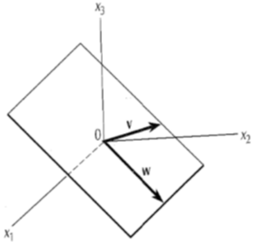
- Let us consider two points $\mathbf{x} = (a, b)$ and $\mathbf{y} = (c, d)$ on line l in the plane. We can obtain the parameterized equation of l as

$$x_2 - b = \frac{d - b}{c - a}(x_1 - a)$$

Parametric Equation

- Let P be a plane on $\mathbb{R}^3$. Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in P. For any scalar s and t, the vector $s\mathbf{v} + t\mathbf{w}$ is called linear combination of $\mathbf{v}$ and $\mathbf{w}$. A parameterization of the plane P is:

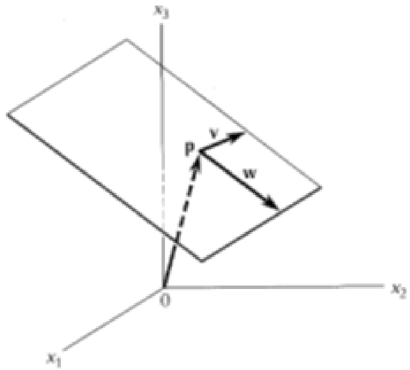
$$\begin{aligned} \mathbf{x} &= s\mathbf{v} + t\mathbf{w} \\ x_1 &= sv_1 + tw_1 \\ x_2 &= sv_2 + tw_2 \\ x_3 &= sv_3 + tw_3 \end{aligned}$$



Parametric Equation

- If the plane does not pass through the origin but through point $\mathbf{p} \neq 0$ and if $\mathbf{v}$ and $\mathbf{w}$ are linearly independent vector from $\mathbf{p}$:

$$\mathbf{x} = \mathbf{p} + s\mathbf{v} + t\mathbf{w}$$



Parametric Equation

- To find the parametric equation of the plane containing the points, **p**, **q** and **r**, we have

$$\begin{aligned}\mathbf{x}(s,t) &= \mathbf{p} + s(\mathbf{q}-\mathbf{p}) + t(\mathbf{r}-\mathbf{P}) \\ &= (1-s-t)\mathbf{p} + s\mathbf{q} + t\mathbf{r}\end{aligned}$$