Question #4 (20 Points). A central planner uses labor L to produce two outputs, x and y, given the following production function, $x = 10L_x - 0.5L_x^2$ and $y = 4L_y - L_y^2$ where L_i is the labor allocated in sector i=x,y. The price in sector x is 2 and the price in sector y is 1. The planner's problem is to maximize aggregate revenue,R:

$$\max_{L_x L_y} R = 2(10L_x - 0.5L_x^2) + (4L_y - L_y^2) \quad s.t.\ \ 16 \geq L_x + L_y;\ L_x > 0, L_y > 0\ .$$

a) How much labor should be allocated in each sector of the economy to achieve the planner's goal? (Hint: show all possible cases)
b) If price in good x rises to 4, how does this affect revenues for the central planner? Show your answer and give a brief intuitive response.

a) Objective of planner

$$\max_{L_x L_y} R = 2(10L_x - 0.5L_x^2) + (4L_y - L_y^2) \quad s.t.\ \ 16 \geq L_x + L_y;\ L_x > 0, L_y > 0$$

Lagrange,

$$\max_{L_2, L_1, \mu} \mathcal{L} = 2(10L_x - 0.5L_x^2) + (4L_y - L_y^2) + \mu \ \ (16 - L_x - L_y)$$

$$\frac{\partial \mathcal{L}}{\partial L_x} = 2(10 - L_x) - \mu \quad = 0$$

$$\frac{\partial \mathcal{L}}{\partial L_y} = \left(4 - 2L_y\right) - \mu \quad = 0$$

$$\mu^{\ *} \frac{\partial \mathcal{L}}{\partial \mu} = \mu^{\ *}(16 - L_x - L_y) = 0$$

$$\mu^{\ *} \geq 0;.\ \ L_x + L_y \leq 16$$

Cases

1. $\mu^{\ *} = 0$,
   Thus, $L_x^* = 10$, $L_y^* = 2$. This yields $R = 2\left(10(10) - 0.5(10)^2\right) + \left(4(2) - (2)^2\right) = 54.$
2. $\mu^{\ *} > 0$,
   Solving, for μ yields the condition $2(10 - L_x) = \left(4 - 2L_y\right)$. This yields $L_x^* = 12$ and $L_y^* = 4$. this yields $R = 2\left(10(12) - 0.5(12)^2\right) + \left(4(4) - (4)^2\right) = 48.$

   Thus the optimal solution is $L_x^* = 10$, $L_y^* = 2$. This yields $R = 104$.

b)Using envelope theorem, $\frac{dR}{dp_x} = 10L_x^* - 0.5L_x^{*2} = 50$. Which leads to an additional revenue of 2*50=100. Note, that the optimal labor value does not change because 10 is the peak of function. The price only augments the function but retains the same maximum.

Question #5 (20 Points). A monopoly supplies its markets from two plants, with the cost function $C(q_1, q_2)$ where q_i is the output of the ith plant. The monopoly faces a linear demand curve $p = a - bQ$ where a >0, b>0 and $Q = q_1 + q_2$.

a) Set up the firm's problem and solve it. Interpret the first order conditions. Show the second order conditions for a maximum. Also, identify all parameters that determine the optimal output in each plant.
b) How does an increase in demand represented by an increase in "a" affect output in plant 1? Prove your answer. Make any assumptions you deem necessary.

a)

The objective of the monopolist is now,

$$\max_{q_1,q_2}(a - b(q_1 + q_2))(q_1 + q_2) - C(q_1, q_2)$$

FOCs

$$a - 2b(q_1 + q_2) - C_1 = 0$$

Marginal revenue from plant 1 output is equal to marginal cost.

$$a - 2b(q_1 + q_2) - C_2 = 0$$

Marginal revenue from plant 1 output is equal to marginal cost.

SOCs

$$\begin{bmatrix} -2b - C_{11} & -2b - C_{12} \\ -2b - C_{21} & -2b - C_{22} \end{bmatrix}$$

The diagonals are negative and the determinant of the Hessian is positive for a maximum.

$$q_i^*(a, b)\ \forall\ i = 1,2$$

b.

Substitute optimal solution into FOCs,

$$a - 2b\big(q_1^*(a, b) + q_2^*(a, b)\big) - C_1(q_1^*(a, b), q_2^*(a, b)) = 0$$

$$a - 2b\big(q_1^*(a, b) + q_2^*(a, b)\big) - C_2(q_1^*(a, b), q_2^*(a, b)) = 0$$

Take derivative with respect to a,

$$1 - 2b\left(\frac{\partial q_1^*}{\partial a} + \frac{\partial q_2^*}{\partial a}\right) - C_{11}\frac{\partial q_1^*}{\partial a} - C_{12}\frac{\partial q_2^*}{\partial a} = 0$$

$$1 - 2b\left(\frac{\partial q_1{}^*}{\partial a} + \frac{\partial q_2{}^*}{\partial a}\right) - C_{21}\frac{\partial q_1{}^*}{\partial a} - C_{22}\frac{\partial q_2{}^*}{\partial a} = 0$$

In matrix form,

$$\begin{bmatrix} -2b - C_{11} & -2b - C_{12} \\ -2b - C_{21} & -2b - C_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial q_1{}^*}{\partial a} \\ \frac{\partial q_2{}^*}{\partial a} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Using Cramer's rule, $\frac{\partial q_1{}^*}{\partial a} = \frac{det\begin{bmatrix} -1 & -2b-C_{12} \\ -1 & -2b-C_{22} \end{bmatrix}}{det\begin{bmatrix} -2b-C_{11} & -2b-C_{12} \\ -2b-C_{21} & -2b-C_{22} \end{bmatrix}} = \frac{+2b+C_{22}-(2b+C_{12})}{(+)} \geq< 0$. This is because it depends on the sign of $C_{22} - C_{12}$. When $C_{22} > (<) C_{12}$, and increase in a leads to more (less) output in plant 1. Note that if they are equal, there is no effect on output since demand for each good increases in the same proportion.