# Recitation 11 (November 9th, 2018)

1. Consider an economy with 2 consumers, Alessandro and Beatrice, $i = \{A, B\}$, one private good $x$, and one public good $G$. Let each consumer have an income of $M$. For simplicity, let the prices of both the public and private good be 1. In addition, the utility functions of consumer $A$ and $B$ are:

$$\begin{aligned} U^A &= \log(x^A) + \log(G), \quad \text{for individual } A, \text{ and} \\ U^B &= \log(x^B) + \log(G), \quad \text{for individual } B \end{aligned}$$

Assume that the public good $G$ is only provided by the contributions of these two individuals, that is, $G = g^A + g^B$.

(a) Find Alessandro's best response function. Depict it in a figure with his contribution, $g^A$, on the vertical axis and Beatrice's contribution, $g^B$, on the horizontal axis.

- The utility maximization problem of Alessandro is that of selecting his consumption of private good, $x$, and his contribution to the public good, $g^A$, to solve

$$\max_{x, g^A} \log x^A + \log G$$

$$\text{subject to } \; x^A + g^A = M \; \text{ and } g^A + g^B = G$$

Taking into account that $x^A = M - g^A$, the above problem can be more compactly expressed as a program with a single choice variable,

$$\max_{g^A} \log(M - g^A) + \log(g^A + g^B)$$

Taking first order condition with respect to $g^A$ yields

$$-\frac{1}{M - g^A} + \frac{1}{g^A + g^B} = 0$$

and solving for $g^A$ we obtain:

$$g^A(g^B) = \frac{M}{2} - \frac{g^B}{2}$$

which represents individual A's best response function (see figure 1).

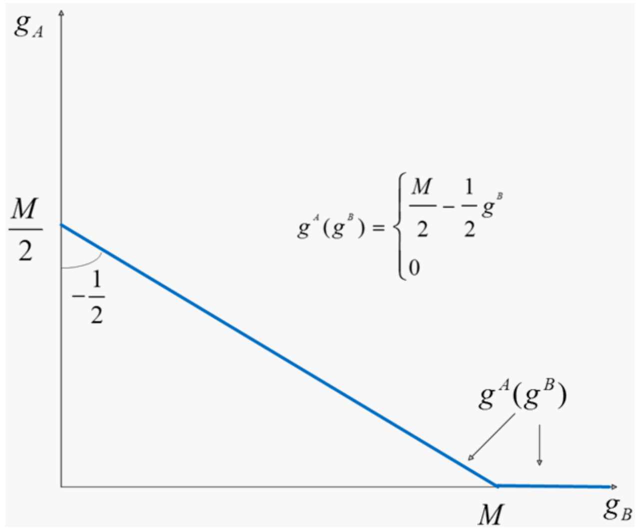


Figure 1. Alessandro's best response function.

- Intuitively, when Beatrice does not contribute to the public good, $g^B = 0$, Alessandro contributes $g^A = \frac{M}{2}$, but as Beatrice increases her contribution (rightward movements in figure 1), Alessandro responds by decreasing his own donation. In the extreme, when Beatrice donates all her wealth to the public good, i.e., $g^B = M$, Alessandro refrains from contributing, $g^A = 0$ for all $g^B \geqslant M$, as represented in the segment of his best response function, $g^A(g^B)$, that overlaps the horizontal axis in the right-hand side of figure 1.

(b) Identify Beatrice's best response function. Depict it in a figure with her contribution, $g^B$, on the horizontal axis and Alessandro's contribution, $g^A$, on the vertical axis.

- Similarly as for Alessandro, the utility maximization decision of Beatrice is that of selecting a contribution to the public good, $g^B$, that solves

$$\max_{g^B} \log(M - g^B) + \log(g^A + g^B)$$

with first order condition

$$-\frac{1}{M - g^B} + \frac{1}{g^A + g^B} = 0$$

solving for $g^B$ yields

$$g^B(g^A) = \frac{M}{2} - \frac{g^A}{2}$$

which represents Beatrice's best response function (see figure 2). Note that we use the same axes as in the best response function of Alessandro, so we can afterwards superimpose both best response functions, $g^A(g^B)$ and $g^B(g^A)$, in the same figure to find their crossing point.

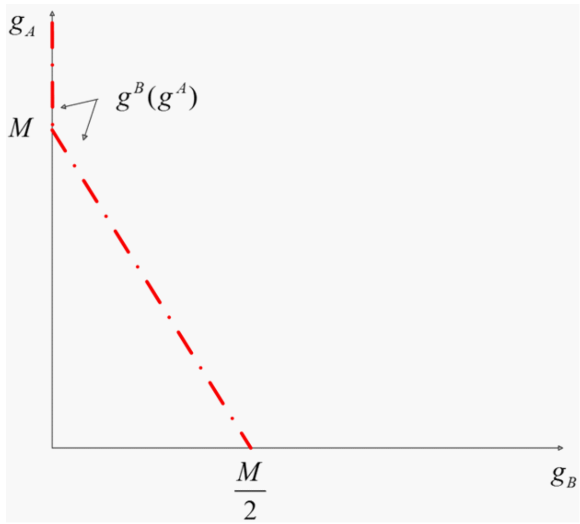


Figure 2. Beatrice's best response function.

(c) *Unregulated equilibrium.* Find the equilibrium contributions to the public good by Alessandro and Beatrice, that is, the Nash equilibrium of this public good game.

- Plugging Beatrice's best response function into Alessandro's best response function,

$$g^A = \frac{M}{2} - \frac{\left(\frac{M}{2} - \frac{g^A}{2}\right)}{2}$$

and solving for $g^A$, we obtain

$$g^A = \frac{M}{3}$$

which identifies the crossing point between both individuals' best response function, as depicted in figure 3. (A similar equilibrium contribution arises for Beatrice, $g^B = \frac{M}{3}$.)

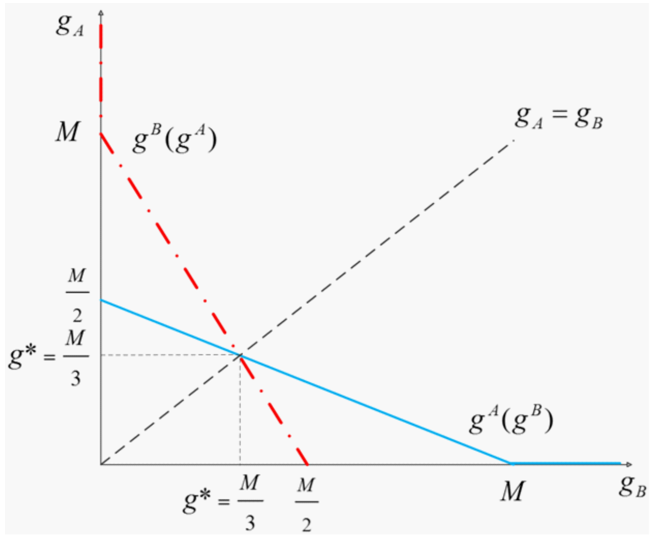


Figure 3. Equilibrium contributions to the public good.

Thus, the aggregate contribution to the public good in the Nash equilibrium is

$$g^A + g^B = \hat{G} = \frac{2M}{3}.$$

(d) *Social optimum.* Find the efficient (socially optimal) contribution to the public good by Alessandro and Beatrice.

- Recall that the utilitarian social welfare is $W = U^A + U^B$. The social planner must therefore choose individual contributions $g^A$ and $g^B$ to solve

$$\max_{g^A, g^B} \log\left(M - g^A\right) + \log(g^A + g^B) + \log\left(M - g^B\right) + \log(g^A + g^B)$$

Since individuals are symmetric, their optimal contributions must coincide, i.e., $g^A = g^B = g$, implying that we can simplify the above problem to

$$\max_{g} \ \log\left(M - g\right) + \log(g + g) + \log\left(M - g\right) + \log(g + g)$$

or

$$\max_{G} 2\log\left(M - g\right) + 2\log(2g)$$

and since $g + g = G$, we can further simplify the above problem to

$$\max_{G} \ 2\log\left(M - \frac{G}{2}\right) + 2\log(G)$$

Taking first order condition with respect to $G$, we obtain

$$-\frac{1}{2}\frac{2}{M-\frac{G}{2}}+\frac{2}{G}=0.$$

Solving for $G$ we find that the optimal aggregate contribution is $\widetilde{G}=M$. Hence, the sum of both individuals' contributions must add up to $M$. In a symmetric outcome this implies that each individual contributes half of this socially optimal level, that is

$$\widetilde{g}^A=\widetilde{g}^B=\frac{M}{2}.$$

- *Comparison.* Comparing $\widetilde{G}=M$ with $\hat{G}=\frac{2M}{3}$ shows that total provision at the Nash equilibrium, where every donor independently selects his own contribution, is below the socially optimal level, i.e., $\hat{G}<\widetilde{G}$.

(e) Use a figure to contrast the Pareto efficient level of private provision and the Nash equilibrium level of provision.

- Figure 4 compares individual contributions in the Nash equilibrium, $\hat{g}=\frac{M}{3}$, and socially optimal (Pareto efficient) contributions, $\frac{M}{2}$, which lie on the middle of the set of allocations satisfying $g^A+g^B=M$.

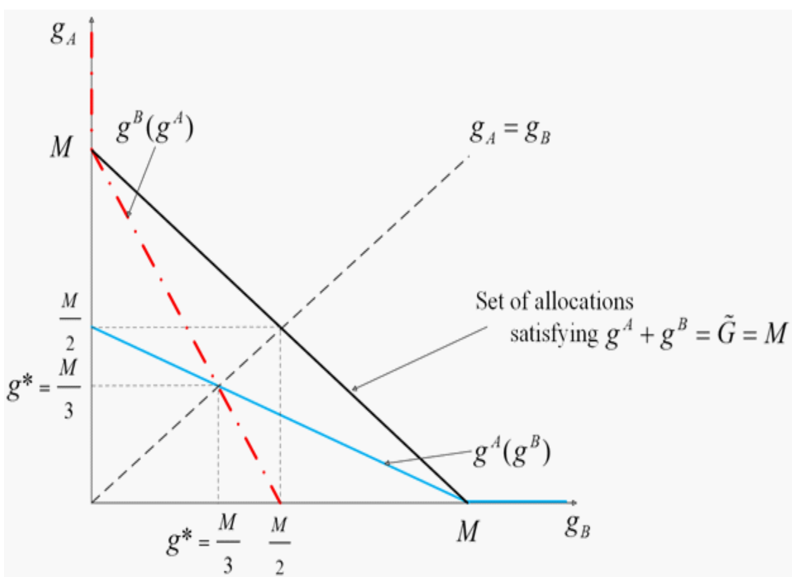


Figure 4. Equilibrium and socially optimal donations.