Recitation 10 (November 2nd, 2018)

1. Consider an economy with two individuals $i = \{1, 2\}$ with the following quasi-linear utility function

$$u_i(s^i, q^i) = v^i(s^i) + \alpha w^i$$

where s^i denotes the speed at which individual i drives his car, w^i is his wealth, and $\alpha > 0$. The utility that individual i obtains from driving fast is $v^i(s^i)$, which is increasing but concave in speed, whereby $\frac{\partial v^i(s^i)}{\partial s^i} > 0$ and $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$. Driving fast, however, increases the probability of suffering a car accident, represented by $\gamma(s^i, s^j)$. This probability is increasing both in the speed at which individual i drives, s^i, and the speed at which other individuals drive, s^j, where $j \neq i$. Hence, the speed of other individuals imposes a negative externality on driver i, since it increases his risk of suffering a car accident. If individual i suffers an accident, he bears a cost of $c^i > 0$, which intuitively embodies the cost of fixing his car, health-care expenses, etc.

(a) *Unregulated equilibrium.* Set up individual i's expected utility maximization problem. Take first-order conditions with respect to s^i, and denote the (implicit) solution to this first-order condition as $\widehat{s}^i$.

- With probability $\gamma(s^i, s^j)$, the individual suffers a car accident, and thus his utility is $v^i(s^i) + \alpha w^i - c^i$, and with probability $1 - \gamma(s^i, s^j)$ he does not suffer the accident, leaving his utility level at $v^i(s^i) + \alpha w^i$.
- Hence, his expected utility is

$$\gamma(s^i, s^j)[v^i(s^i) + \alpha w^i - c^i] + (1 - \gamma(s^i, s^j))[v^i(s^i) - \alpha w^i],$$

which reduces to $v^i(s^i) + \alpha w^i - \gamma(s^i, s^j)c^i$. Hence, every individual i maximizes his expected utility by choosing an speed level s^i that solves

$$\max_{s^i} \; v^i(s^i) + \alpha w^i - \gamma(s^i, s^j) \times c^i$$

Taking first-order conditions with respect to s^i we obtain

$$\frac{\partial v^i(s^i)}{\partial s^i} - \frac{\partial \gamma}{\partial s^i} c^i = 0 \tag{1}$$

Hence, driver i independently selects the speed, $\widehat{s}^i$, that solves $\frac{\partial v^i(s^i)}{\partial s^i} = \frac{\partial \gamma}{\partial s^i} c^i$.

- Intuitively, driver i increases his speed s^i until the point where the additional utility from marginally increasing s^i, $\frac{\partial v^i(s^i)}{\partial s^i}$, coincides with its associated expected individual cost from speed, i.e., a higher probability of suffering a car accident times its associated cost, as measured by $\frac{\partial \gamma}{\partial s^i} c^i$.

- *Parametric example.* Consider, for instance, a utility from driving fast of $v(s^i) = \sqrt{s^i}$ (which is increasing and concave in s^i, as required), and that the probability of suffering a car accident is $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$, where $\beta_i > \beta_j$ (indicating that my own speed increases the probability that I suffer a car accident more than other drivers' speeds). First order condition (1) in this context becomes

$$\frac{1}{2\sqrt{s^i}} = \beta_i c^i,$$

and solving for s^i, we obtain an equilibrium speed of $\widehat{s^i} = \frac{1}{4(\beta_i c^i)^2}$ for every individual driver $i = \{1, 2\}$.

(b) *Social optimum.* Set up the social planner's expected welfare maximization problem. Take first-order conditions with respect to s^1 and s^2. Denote the (implicit) solution to this first-order condition as $\overline{s}^i$.

- The social planner solves the expected welfare maximization problem

$$\max_{s^1, s^2} \; v^1(s^1) + \alpha w^1 + v^2(s^2) + \alpha w^2 - \gamma(s^1, s^2) \times \left(c^1 + c^2\right)$$

Taking first-order conditions with respect to s^1, we obtain that $\bar{s}^1$ solves

$$\frac{\partial v^1(s^1)}{\partial s^1} = \frac{\partial \gamma}{\partial s^1} \left(c^1 + c^2\right) \tag{2}$$

and similarly with respect to s^2, we obtain that $\bar{s}^2$ solves

$$\frac{\partial v^2(s^2)}{\partial s^2} = \frac{\partial \gamma}{\partial s^2} \left(c^1 + c^2\right) \tag{3}$$

Intuitively, at the social optimum every driver i increases his speed s^i until the point where the additional utility from marginally increasing s^i coincides with its associated expected social cost from speed, measured by not only the higher probability of him suffering a car accident but also by the higher probability that the other individual $j \neq i$ suffers a car accident because of the speed s^i of individual i.

(c) *Comparison.* Show that drivers have individual incentives to drive too fast, relative to the socially optimal speed, i.e., show that $\widehat{s^i} > \overline{s}^i$.

- Comparing expressions (1) and (2), yields

$$\frac{\partial v^1(\hat{s}^1)}{\partial s^1} < \frac{\partial v^1(\bar{s}^1)}{\partial s^1}$$

Since $\frac{\partial^2 v^i(s^i)}{(\partial s^i)^2} < 0$ by definition, $\frac{\partial v^i(s^i)}{\partial s^i}$ is a decreasing function. Therefore, the speed that individual 1 independently selects, $\widehat{s}^1$, is excessive from a social point of view, i.e., $\widehat{s}^1 > \overline{s}^1$. Similarly, comparing (1) and (3), we have that $\widehat{s}^2 > \overline{s}^2$. Intuitively, every driver does not internalize the negative externality that his speed imposes on other drivers (in the form of a higher probability of suffering a car accident) when he independently selects his driving speed.

- Figure 1 represents the marginal utility, $\frac{\partial v^i(s^i)}{\partial s^i}$, and marginal expected costs, individual marginal costs, $\frac{\partial \gamma}{\partial s^i} c^i$, and social marginal costs, $\frac{\partial \gamma}{\partial s^i}(c^i + c^j)$, to support the above explanation. Since the social marginal cost curve is higher for any speed level s^i than the individual marginal cost curve, the former crosses the marginal utility curve at a lower speed level, i.e., $\overline{s}^i < \widehat{s}^i$. Intuitively, the social planner internalizes the externality that additional speed imposes on other drivers (who could suffer a car accident due to the speed of driver i), and thus reduces the speed of both drivers.

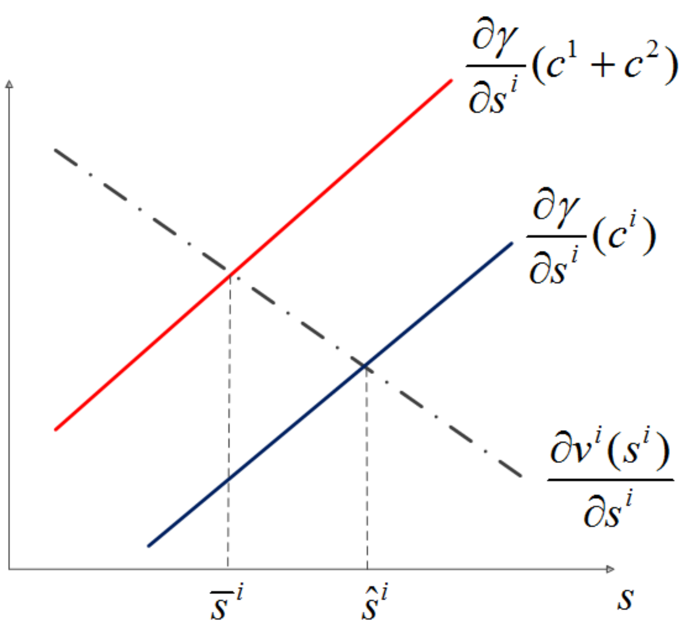

Figure 1. Efficient and socially optimal speed.

  - Note that for simplicity, we consider that the marginal utility decreases in s^i at a constant rate, i.e., $\frac{\partial^2 v^i(s^i)}{\partial {s^i}^2}$ is constant in s^i or, alternatively, $\frac{\partial^3 v^i(s^i)}{(\partial s^i)^3} = 0$; implying that the marginal utility curve is a straight line. In addition, we also assume that further increases in speed s^i imply a constant increase in the probability of an accident, i.e., $\frac{\partial^2 \gamma}{(\partial s^i)^2} > 0$ but constant or, alternatively, that $\frac{\partial^3 \gamma}{(\partial s^i)^3} = 0$. This property entails the marginal cost curve is also a straight line.

– *Parametric example.* Continuing with the previous example in which $v^i(s^i) = \sqrt{s^i}$ and $\gamma(s^i, s^j) = \beta_i s^i + \beta_j s^j$, the socially optimal speed that the social planner would select, $\overline{s}^i$, is that satisfying

$$\frac{1}{2\sqrt{\overline{s}^i}} = \beta_i(c^i + c^j)$$

and solving for $\overline{s}^i$ yields $\overline{s}^i = \frac{1}{4[\beta_i(c^i+c^j)]^2}$, which clearly falls below the speed level independently selected by every driver $\widehat{s}^i = \frac{1}{4(\beta_i c^i)^2}$.

(d) *Restoring the social optimum.* Let us now evaluate the effect of speeding tickets (fines) to individuals driving too fast, i.e., to those drivers with a speed $\widehat{s}^i$ satisfying, $\widehat{s}^i > \overline{s}^i$. What is the dollar amount of the fine m^i that induces every individual i to fully internalize the externality he imposes onto others?

- Comparing (1) and (2) for driver 1, we must impose a fine of $m^1 = c^2$ in order to guarantee that (1) coincides with (2). Intuitively, this fine induces driver 1 to internalize the negative externality (higher chances of suffering a car accident and, in this case, an associated monetary cost of repairs) that he imposes on driver 2. Similarly comparing (1) and (3) for driver 2, we must impose a fine of $m^2 = c^1$ in order to guarantee that (1) coincides with (3).

(e) Let us now consider that individuals obtain a utility from driving fast, $v^i(s^i)$, only in the case that no accident occurs. Repeat steps (a)-(c), finding the optimal fine m^i that induces individuals to fully internalize the externality.

- *Equilibrium speed.* In this section of the exercise, driver i only obtains utility from driving fast, $v^i(s^i)$, when no accident occurs. Given that the probability that an accident does *not* occur is $1 - \gamma(s^1, s^2)$, the utility of driver i is

$$\underbrace{\left[1 - \gamma(s^1, s^2)\right]\left(v^i(s^i) + \alpha w^i\right)}_{\text{No accident}} + \underbrace{\gamma(s^i, s^j)(\alpha w^i - c^i)}_{\text{Accident}}$$

which can be rearranged as

$$v^i(s^i) + \alpha w^i - \gamma(s^i, s^j)\left[c^i + v^i(s^i)\right]$$

Taking first order conditions with respect to s^i, we obtain that the individual driver i independently selects the speed $\overline{s}^i$ that solves

$$\frac{\partial v^i(s^i)}{\partial s^i}\left[1-\gamma(s^i,s^j)\right]=\frac{\partial \gamma}{\partial s^i}(c^i+v^i(s^i)) \tag{4}$$

where conveniently separates the marginal utility of driving faster in the left-hand side, which only arises if driver i does not suffer a car accident, an event with probability $1-\gamma(s^i,s^j)$; and its associated marginal cost in the right-hand side, which captures the higher probability of suffering a car accident, $\frac{\partial \gamma}{\partial s^i}$, and its two costs: one explicit, c^i, and one implicit, namely, the utility from driving that driver i would have to give up (since he can only benefit from driving when he does not suffer a car accident).

- *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

  $$\frac{1}{2\sqrt{\overline{s}^i}}\left[1-(\beta_i\overline{s}^i+\beta_j\overline{s}^j)\right]=\beta_i(c^i+\sqrt{\overline{s}^i}),$$

  and similarly for driver j. Before solving for $\overline{s}^i$ in order to driver i's best response function, let us assume (in order to keep our parametric example compact) that $\beta_i=\beta_j=\frac{1}{2}$ and $c^i=c^j=\frac{2}{3}$. In this context, solving for $\overline{s}^i$ we obtain

  $$\overline{s}^i(\overline{s}^j)=2-3\overline{s}^j-\frac{4}{3}\sqrt{\overline{s}^j}$$

  Since both drivers are symmetric, $\overline{s}^i=\overline{s}^j$, we can solve for $\overline{s}^i$ yielding a symmetric equilibrium speed level of $\overline{s}^i=0.313$.

- *Socially optimal speed.* The social planner's maximization problem in this case becomes

$$\max_{s^1,s^2}\ v^1(s^1)+\alpha w^1-\gamma(s^1,s^2)\left[c^1+v^1(s^1)\right]+v^2(s^2)+\alpha w^2-\gamma(s^1,s^2)\left[c^2+v^2(s^2)\right]$$

Taking first order conditions with respect to s^i, we obtain that the socially optimal speed, $\widehat{s}^i$, solves

$$\frac{\partial v^i(s^i)}{\partial s^i}\left[1-\gamma(s^i,s^j)\right]=\frac{\partial \gamma}{\partial s^i}\left[c^i+v^i(s^i)\right]+\frac{\partial \gamma}{\partial s^i}\left[c^j+v^j(s^j)\right] \tag{5}$$

- *Comparison.* Comparing expressions (4) and (5), we obtain that the fine m^i that induces every individual i to internalize the externality that his driving imposes on others is

$$m^i=c^j+v^j(s^j)$$

Intuitively, now an increase in the speed of driver i not only increases the probability that driver j suffers a car accident, and thus needs to incur a cost of c^j, it also reduces the utility from driving that driver j can only experience if he is not involved in a car accident.

– *Parametric example.* Following with the on-going parametric example, the above first order condition (4) becomes

$$\frac{1}{2\sqrt{\widehat{s}^{ii}}}\left[1-(\beta_i\widehat{s}^i+\beta_j\widehat{s}^j)\right]=\beta_i\left[(c^i+\sqrt{\widehat{s}^i})+(c^j+\sqrt{\widehat{s}^j})\right],$$

Before solving for $\widehat{s}^i$ in order to driver i's best response function, let us assume (in order to keep our parametric example compact) that $\beta_i = \beta_j = \frac{1}{2}$ and $c^i = c^j = \frac{2}{3}$. In this context, we can simultaneously solve for $\widehat{s}^i$ and $\widehat{s}^j$ obtaining $\widehat{s}^i = \widehat{s}^j = 0.157$, which is indeed a lower speed than when drivers independently choose their own driving speed, $\overline{s}^i = 0.313$.