EconS 501 Midterm #2 - November 14th, 2018

Show all your work clearly and make sure you justify all your answers.
NAME__________________________

1. Consider an industry where two firms simultaneously and independently choose how much to recycle. Firm i's marginal cost from recycling is c_i while firm j's marginal cost is c_j, where $c_i \geq c_j > 0$. Firm i receives a benefit from recycling equal to

$$B_i(r_i, r_j) = a - r_i + \beta_i r_j,$$

where r_i denotes firm i's recycling output, r_j indicates firm j's recycling output. In addition, parameter a satisfies $a > c_i, c_j$ and $\beta_i \in [0, 1]$. Intuitively, parameter β_i measures how much firm j's recycling decisions benefits firm i (a positive externality). That is, when $\beta_i = 1$ firm i fully benefits from every unit of firm j's recycling activity, whereas when $\beta_i = 0$ firm j's recycling does not produce any benefit on firm i. Firm j's benefit from recycling is symmetric, that is, $B_j(r_i, r_j) = a - r_j + \beta_j r_i$.

(a) Set up every firm i's profit-maximization problem, find its best response function $r_i(r_j)$, and discuss whether firms recycling activities are strategic substitutes or complements.

- Every firm i solves

$$\max_{r_i \geq 0} \ \pi_i(r_i, r_j) = (a - r_i + \beta_i r_j) r_i - c_i r_i$$

Differentiating with respect to its recycling output r_i, and solving for r_i, we find a best response function

$$r_i(r_j) = \frac{a - c_i}{2} + \frac{\beta_i}{2} r_j$$

Graphically, this best response function originates at $\frac{a-c_i}{2}$ and increases in firm j's recycling r_j at a rate of $\frac{\beta_i}{2}$, indicating that firms' recycling activities are strategic complements. A symmetric argument for firm j yields best response function

$$r_j(r_i) = \frac{a - c_j}{2} + \frac{\beta_j}{2} r_i$$

which is also increasing in its rival's recycling activities, r_i.

(b) Identify the equilibrium level of recycling every firm selects, i.e., r_i^* and r_j^*, respectively. How are your equilibrium results affected by costs c_i and c_j? How are they affected by parameters β_i and β_j?

- Simultaneously solving the two best response functions yields equilibrium recycling levels of

$$r_i^* = \frac{2(a - c_i) + \beta_i(a - c_j)}{4 - \beta_i \beta_j} \quad \text{and} \quad r_j^* = \frac{2(a - c_j) + \beta_j(a - c_i)}{4 - \beta_i \beta_j}$$

Note that firm i's equilibrium recycling r_i^* is positive as long as $c_i < a + \frac{\beta_i(a-c_j)}{2}$. As a curiosity, note that when positive externalities are absent, $\beta_i = \beta_j = 0$, this condition on c_i simplifies to $c_i < a$ as in similar simultaneous-move games without externalities; whereas when positive externalities are maximal, $\beta_i = \beta_j = 1$, the condition on c_i becomes $c_i < a + \frac{(a-c_j)}{2}$. Since the right-hand side of this condition on c_i increases in β_i, firm i is more willing to choose positive recycling amounts as positive externalities it receives from firm j become more substantial.

- *Symmetric case.* When costs are symmetric, $c_i = c_j = c$, and positive externalities are symmetric, $\beta_i = \beta_j = \beta$, the above equilibrium recycling amounts simplify to

$$r_i^* = r_j^* = \frac{2(a-c) + \beta(a-c)}{4 - \beta\beta} = \frac{(2+\beta)(a-c)}{(2+\beta)(2-\beta)} = \frac{a-c}{2-\beta}$$

- *Comparative statics - Costs.* Note that

$$\begin{aligned} \frac{\partial r_i^*}{\partial c_i} &= -\frac{2}{4-\beta_i\beta_j} < 0 \text{ and} \\ \frac{\partial r_i^*}{\partial c_j} &= -\frac{\beta_j}{4-\beta_i\beta_j} < 0, \end{aligned}$$

indicating that if firm i or its competitor become less efficient (cost c_i or c_j increases) firm i responds reducing its equilibrium recycling.

- *Comparative statics - Positive externality.* In addition,

$$\begin{aligned} \frac{\partial r_i^*}{\partial \beta_i} &= \frac{2\beta_j(a-c_i) + 4(a-c_j)}{\left(4-\beta_i\beta_j\right)^2} > 0, \text{ and} \\ \frac{\partial r_i^*}{\partial \beta_j} &= \frac{\beta_i\left[2(a-c_i) + 4(a-c_j)\right]}{\left(4-\beta_i\beta_j\right)^2} > 0 \end{aligned}$$

suggesting that if the positive externality becomes stronger firms respond increasing their recycling. However, firm i responds more significantly to an increase in the positive externality it receives from its rival, β_i, than when its rival receives a larger externality from firm i, β_j.

(c) *Social optimum.* Assume that social welfare only considers the sum of every firms' profits. Identify the socially optimal levels of recycling, i.e., r_i^{SO} and r_j^{SO}.

- A social planner considering both firms' profits (or a partnership between both firms that considers both of their profits) selects the recycling level for both firms, (r_i, r_j), that solves

$$\begin{aligned} &\max_{r_i, r_j} \pi_i(r_i, r_j) + \pi_j(r_i, r_j) \\ = &\left[(a - r_i + \beta_i r_j) r_i - c_i r_i\right] + \left[\left(a - r_j + \beta_j r_i\right) r_j - c_j r_j\right] \end{aligned}$$

Taking first-order conditions with respect to r_i and r_j yields, respectively,

$$\begin{aligned} a - 2r_i + \beta_i r_j - c_i + \beta_j r_j &= 0, \text{ and} \\ a - 2r_j + \beta_j r_i - c_j + \beta_i r_i &= 0. \end{aligned}$$

Simultaneously solving for r_i and r_j in the above two first-order conditions yields the socially optimal levels of recycling

$$r_i^{SO} = \frac{2(a-c_i) + \left(\beta_i + \beta_j\right)(a-c_j)}{\left(4-\beta_i\beta_j\right)\left(2+\beta_i+\beta_j\right)}$$

for every firm i.

- *Symmetric case.* When costs are symmetric, $c_i = c_j = c$, and positive externalities are symmetric, $\beta_i = \beta_j = \beta$, the above socially optimal recycling simplifies to

$$r_i^{SO} = r_j^{SO} = \frac{2(a-c) + (\beta+\beta)(a-c)}{(4-\beta\beta)(2+\beta+\beta)} = \frac{a-c}{2-2\beta}$$

(d) *Comparison.* Compare the equilibrium recycling amounts that you found in part (b) and at the social optimum (from part c). Interpret your findings.

- Comparing r_i^* and r_i^{SO}, we find that $r_i^* < r_i^{SO}$ since

$$\frac{2(a-c_i)+\beta_i(a-c_j)}{4-\beta_i\beta_j} < \frac{2(a-c_i) + \left(\beta_i+\beta_j\right)(a-c_j)}{\left(4-\beta_i\beta_j\right)\left(2+\beta_i+\beta_j\right)}$$

simplifies to

$$0 < \frac{\beta_j(a-c_j)}{2+\beta_i+\beta_j}.$$

Intuitively, when every firm independently chooses its own recycling amount, it ignores the positive externality that this recycling produces on its rival. However, the social planner internalizes this externality, thus recommending a larger amount of recycling.

(e) *Numerical example.* Evaluate your results in parts (b) and (c) assuming parameter values $c_i = c_j = c$ and $\beta_i = \beta_j = \frac{1}{2}$. Find social welfare in equilibrium recycling amounts (from part b) and at the social optimum (from part c). Compare these two social welfares, and discuss your results.

- Social welfare evaluated at the equilibrium we found in part (a) is

$$SW^* = \pi_i\left(r_i^*, r_j^*\right) + \pi_j\left(r_i^*, r_j^*\right)$$

Since $c_i = c_j = c$ and $\beta_i = \beta_j = \frac{1}{2}$, equilibrium recycling simplifies to $r_i^* = r_j^* = \frac{2(a-c)}{3}$ and equilibrium profits become

$$\begin{aligned}
\pi_i\left(r_i^*, r_j^*\right) &= \pi_j\left(r_i^*, r_j^*\right) = \left(a - \frac{2(a-c)}{3} + \frac{2(a-c)}{6}\right)\frac{2(a-c)}{3} - c\frac{2(a-c)}{3} \\
&= \frac{2(a-c)}{3}\left[a - \frac{2(a-c)}{6} - c\right] \\
&= \left[\frac{2(a-c)}{3}\right]^2 = \frac{8(a-c)^2}{9}
\end{aligned}$$

Hence, social welfare in equilibrium becomes

$$SW^* = \pi_i\left(r_i^*, r_j^*\right) + \pi_j\left(r_i^*, r_j^*\right) = 2\left[\frac{2(a-c)}{3}\right]^2$$

- Social welfare at the social optimum found in part (b) is

$$SW^{SO} = \pi_i\left(r_i^{SO}, r_j^{SO}\right) + \pi_j\left(r_i^{SO}, r_j^{SO}\right)$$

Since $c_i = c_j = c$ and$\beta_i = \beta_j = \frac{1}{2}$, socially optimal recycling simplifies to $r_i^{SO} = r_j^{SO} = a - c$ and profits become

$$\begin{aligned}
\pi_i\left(r_i^{SO}, r_j^{SO}\right) &= \left[\left(a - (a-c) + \frac{1}{2}(a-c)\right)(a-c) - c(a-c)\right] \\
&= (a-c)\left[\frac{1}{2}a + \frac{1}{2}c - c\right] \\
&= (a-c)\left(\frac{1}{2}a - \frac{1}{2}c\right) = \frac{1}{2}(a-c)^2.
\end{aligned}$$

entailing that social welfare reduces to

$$SW^{SO} = \pi_i\left(r_i^{SO}, r_j^{SO}\right) + \pi_j\left(r_i^{SO}, r_j^{SO}\right) = (a-c)^2$$

- Comparing SW^* and SW^{SO}, it is clear that $SW^{SO} > SW$ since $(a-c)^2 > \frac{8(a-c)^2}{9}$. That is, the social planner selects a recycling output pair that generates higher social welfare than each firm independently.

2. Consider a setting in which a worker can exert two levels of efforts, e_1 and e_2, where $0 \leq e_i \leq +\infty$ for every $i \in \{1, 2\}$. Intuitively, she may spend some hours a day assembling products, some time presenting them to potential customers, and finally cleaning the store, or keeping records of all costs and receipts. For simplicity, assume that output y_1 (y_2) is only generated from effort e_1 (e_2), and let g_1 (g_2) denote the agent's efficiency at producing output 1 (2, respectively). Each output behaves as follows,

$$y_1 = g_1 e_1 + \varepsilon_1 \text{ and } y_2 = g_2 e_2 + \varepsilon_2$$

Random shocks ε_1 and ε_2, which affect outputs 1 and 2 respectively, follow a bivariate normal distribution $N(\mathbf{0}, \Sigma)$, with expectation 0 for both outputs and variance-covariance matrix of

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

The agent earns a wage of $w = s_1 y_1 + s_2 y_2$, where s_1 and s_2 represent his output shares, with $0 \leq s_i \leq 1$ for $i \in \{1, 2\}$. Cost is increasing and convex in both effort levels, where

$$c(e_1, e_2) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2$$

such that her payoff function now becomes

$$U = u(w) - c(e_1, e_2)$$

and her utility function is $u(w) = 1 - \exp(-\eta w)$.

(a) Find the certainty equivalent payment of the agent.

- The agent's expected utility from earning wage w is

$$E_A\left[u\left(w\right)\right] = 1 - \int_{-\infty}^{+\infty} \exp\left(-\eta w\right) f\left(w\right) dw$$

where $w = s_1 y_1 + s_2 y_2$ following a normal distribution of $\mathcal{N}\left(\mu_w, \sigma_w^2\right)$. Specifically, the agent's expected wage, μ_w, is given by

$$\begin{aligned}
\mu_w &= E\left[w\right] \\
&= E\left[s_1 \underbrace{\left(g_1 e_1 + \varepsilon_1\right)}_{y_1} + s_2 \underbrace{\left(g_2 e_2 + \varepsilon_2\right)}_{y_2}\right] \\
&= s_1 g_1 e_1 + s_1 \underbrace{E\left[\varepsilon_1\right]}_{0} + s_2 g_2 e_2 + s_2 \underbrace{E\left[\varepsilon_2\right]}_{0} \\
&= s_1 g_1 e_1 + s_2 g_2 e_2
\end{aligned}$$

and the variance of her wage, σ_w^2, is given by

$$\begin{aligned}
\sigma_w^2 &= Var\left(w\right) \\
&= Var\left(s_1\left(g_1 e_1 + \varepsilon_1\right) + s_2\left(g_2 e_2 + \varepsilon_2\right)\right) \\
&= Var\left(s_1 \varepsilon_1 + s_2 \varepsilon_2\right) \\
&= \begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \\
&= s_1^2 \sigma_1^2 + 2 s_1 s_2 \sigma_{12} + s_2^2 \sigma_2^2
\end{aligned}$$

- Since both μ_w and σ_w^2 are scalar terms (as the agent's wage is a linear combination of her output shares), her wage w follows a univariate normal distribution of

$$f\left(w\right) = \frac{1}{\sqrt{2\pi}\sigma_w} \exp\left(-\frac{1}{2}\left(\frac{w - \mu_w}{\sigma_w}\right)^2\right)$$

that is the same as part (a), entailing a certainty equivalent payment of

$$\begin{aligned}
CE\left(e\right) &= \mu_w - \eta \frac{\sigma_w^2}{2} \\
&= \underbrace{s_1 g_1 e_1 + s_2 g_2 e_2}_{\mu_w} - \frac{\eta}{2} \underbrace{\left(s_1^2 \sigma_1^2 + 2 s_1 s_2 \sigma_{12} + s_2^2 \sigma_2^2\right)}_{\sigma_w^2}
\end{aligned}$$

(b) How is the agent's certainty equivalent affected when she becomes more risk averse?

- Differentiating the certainty equivalent payment with respect to η, we find

$$\frac{\partial CE\left(e\right)}{\partial \eta} = -\frac{1}{2}\left(s_1^2 \sigma_1^2 + 2 s_1 s_2 \sigma_{12} + s_2^2 \sigma_2^2\right) < 0$$

such that when the agent becomes more risk averse, she demands a lower certainty equivalent payment. Intuitively, as the agent exhibits lower tolerance for output shocks, she is willing to accept a lower sure payment (that is, a higher risk premium) to make her indifferent to receiving volatile wages.

(c) How is the agent's certainty equivalent affected when output shocks become more volatile?

- Differentiating the certainty equivalent payment with respect to σ_1, σ_2, and σ_{12}, we obtain

$$\frac{\partial CE\,(e)}{\partial \sigma_1} = -\eta s_1^2 \sigma_1 < 0$$
$$\frac{\partial CE\,(e)}{\partial \sigma_2} = -\eta s_2^2 \sigma_2 < 0$$

such that when output shocks become volatile (that is, σ_1 or σ_2 increases in magnitude), the risk-averse agent demands a lower certainty equivalent payment.

3. Consider two consumers with utility functions over two goods, x_1 and x_2, given by

$$\begin{aligned} u_A &= \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) \quad \text{for consumer } A\text{, and} \\ u_B &= \log(x_1^B) + x_2^B - \frac{1}{2}\log(x_1^A) \quad \text{for consumer } B. \end{aligned}$$

where the consumption of good 1 by individual $i = \{A, B\}$ creates a negative externality on individual $j \neq i$ (see the third term, which enters negatively on each individual's utility function). For simplicity, consider that both individuals have the same wealth, m, and that the price for both goods is 1.

(a) *Unregulated equilibrium.* Set up consumer A's utility maximization problem, and determine his demand for goods 1 and 2, as x_1^A and x_2^A. Then operate similarly to find consumer B's demand for good 1 and 2, as x_1^B and x_2^B.

- Consumer A chooses x_1^A and x_2^A to solve

$$\max_{(x_1^A, x_2^A)} \ \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to} \ \ x_1^A + x_2^A = M$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M - x_1^A - x_2^A),$$

which yields first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - x_1^A - x_2^A = 0$$

Solving for x_1^A, we obtain $\frac{1}{x_1^A} = 1$, i.e., $x_1^A = 1$, which implies $M - 1 - x_2^A = 0$, or $x_2^A = M - 1$. Hence, consumer A's optimal consumption is

$$x_1^A = 1 \text{ and } x_2^A = M - 1$$

A similar argument applies to consumer B,

$$x_1^B = 1 \text{ and } x_2^B = M - 1$$

(b) *Social optimum.* Calculate the socially optimal amounts of x_1^A, x_2^A, x_1^B and x_2^B, considering that the social planner maximizes a utilitarian social welfare function, namely, $W = U_A + U_B$.

- The socially optimal consumption in this case solves

$$\max_{(x_1^A, x_2^A)} U^A + U^B \quad \text{subject to } x_1^A + x_2^A = M \text{ and } x_1^B + x_2^B = M$$

The Lagrangian for this social planner's problem is

$$\mathcal{L} = \frac{1}{2}\log(x_1^A) + \frac{1}{2}\log(x_1^B) + x_2^A + x_2^B + \lambda^A(M - x_1^A - x_2^A) + \lambda^B(M - x_1^B - x_2^B)$$

Taking first-order conditions, we find the socially optimal consumption profile:

$$x_1^A = \frac{1}{2} \text{ and } x_2^A = M - \frac{1}{2}$$

$$x_1^B = \frac{1}{2} \text{ and } x_2^B = M - \frac{1}{2}$$

Intuitively, the social planner recommends a lower consumption of good 1 (the good that generates the negative externality), and an increase in the consumption of good 2, for both individuals.

(c) *Restoring efficiency.* Show that the social optimum you found in part (b) can be induced by a tax on good 1 (so the after-tax price becomes $1+t$) with the revenue returned equally to both consumers in a lump-sum transfer.

- With tax t^A placed on good 1 and with lump-sum transfer T^A, consumer A solves

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to } (1 + t^A)x_1^A + x_2^A = M + T^A$$

where note that the price of good 1 increased from 1 to $(1 + t^A)$, but this consumer also sees his wealth increase by the lump sum T^A. The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M + T^A - (1 + t^A)x_1^A - x_2^A)$$

Taking first-order conditions, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A(1+t^A) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M + T^A - (1+t^A)x_1^A - x_2^A = 0$$

Simultaneously solving for x_1^A and x_2^A, we find that consumer A's consumption bundles after introducing the tax become

$$x_1^A = \frac{1}{1+t^A} \text{ and } x_2^A = M + T^A - 1$$

Similarly we find the optimal consumption of consumer B who pays tax t^Bon good 1 and receives T^Bas a lump-sum transfer:

$$x_1^B = \frac{1}{1+t^B} \text{ and } x_2^B = M + T^B - 1$$

- *Comparison.* Comparing the optimal consumption levels found in part (b) with the equilibrium outcomes found in part (c), the tax imposed on any individual $i = A, B$ must hence satisfy

$$\frac{1}{2} = \frac{1}{1+t^i},$$

which would guarantee that equilibrium and socially optimal amounts coincide. Solving for the tax t^i yields $t^i = \$1$. Hence, by setting a tax of $t^i = \$1$ on the consumption of good 1, and returning the tax revenue to this individual in a lump-sum transfer, efficiency is restored, yielding a consumption

$$x_1^i = \frac{1}{1+1} = \frac{1}{2} \text{ of good 1,}$$

and

$$\begin{aligned} x_2^i &= M + T^i - 1 \\ &= M + \frac{1}{2} - 1 = M - \frac{1}{2} \text{ of good 2,} \end{aligned}$$

as described in the socially optimal amounts found in part (b).

4. Consider a perfectly competitive industry with N symmetric firms, each with cost function $c(q) = F + cq$, where $F, c > 0$. Assume that the inverse demand is given by $p(Q) = a - bQ$, where $a > c$, $b > 0$, and where Q denotes aggregate output.

   (a) *Short-run equilibrium.* If exit and entry is not possible in the industry (assuming N firms remain active), find the individual production level of each firm.

- Each individual firm i solves the PMP

$$\max_{q_i \geq 0} (a - bQ)q_i - (F + cq_i) = \left(a - bq_i - b \sum_{j \neq i} q_j \right) q_i - (F + cq_i)$$

Taking first-order conditions with respect to q_i yields

$$a - 2bq_i - b \sum_{j \neq i} q_j - c = 0$$

and applying symmetry in equilibrium outputs, i.e., $q_1 = q_2 = ... = q_N$, we obtain an individual equilibrium output

$$q_i = \frac{a - c}{b(N + 1)}$$

for every firm $i \in N$. Note that this result is a function of the number of active firms in the industry, N.

- In this setting, the equilibrium market price is

$$p^* = a - b \underbrace{\left(N \cdot \frac{a - c}{b(N + 1)} \right)}_{Q = N \cdot q_i} = \frac{a + Nc}{N + 1}$$

(b) *Long-run equilibrium.* Consider now that firms have enough time to enter the industry (if economic profits can be made) or to exit (if they make losses by staying in the industry). Find the long-run equilibrium number of firms in this perfectly competitive market.

- In a long-run equilibrium of a perfectly competitive market, we know that firms must be making no economic profits, $\pi = 0$, as otherwise firms would still have incentives to enter or exit the industry. Hence, we first need to find the equilibrium profits that every individual firm i earns by producing the equilibrium output q_i found in part (a). In particular, these profits are

$$\pi_i = (a - b \underbrace{Nq_i}_{Q})q_i - (F + cq_i) = \frac{(a - c)^2}{b(N + 1)^2} - F$$

setting them equal to zero and solving for N yields the long-run equilibrium number of firms, $\lfloor N^* \rfloor = \frac{a-c}{\sqrt{bF}} - 1$, where $\lfloor N \rfloor$ indicates the highest integer smaller or equal to N. For instance, if $a = b = 1$, $c = \frac{1}{4}$, and $F = \frac{1}{16}$ N^* becomes $N^* = 2$.

- More generally, note that the expression we found for equilibrium profits, π_i, is monotonically decreasing in the number of firms, N, for all parameter values, that is

$$\frac{\partial \pi_i}{\partial N} = -2b(a - c)^2(N - 1) < 0$$

thus implying that equilibrium profits becomes zero for a sufficiently large number of firms.