Homework # 8 - [Due on November 16th, 2018]

1. Consider an industry with two polluting firms, but where one of them produces more environmental damage per unit of output (labelled as the brown firm) than its rival (green firm). Firm i's marginal production costs are given by c_i where $i = \{B, G\}$, where c_B is allowed to be higher or lower than c_G. In addition, the social welfare function that the regulator uses to set emission fees on these firms is

$$SW = CS + PS + T - Env$$

where CS is the consumer surplus, PS is the producer surplus, $T = t_B q_B + t_G q_G$ is the tax revenue from emission fees on both firms, and $Env = d_B(q_B)^2 + d_G(q_G)^2$ is the environmental damage from the production of the brown and green firms, where $d_B \geq d_G > \frac{1}{2}$. Finally, assume that firms sell differentiated products. In particular, the inverse demand function of firm $i = \{B, G\}$ is

$$p_i(q_i, q_j) = 1 - q_i - gq_j \text{ where } j = \{B, G\} \text{ and } j \neq i.$$

Hence, parameter $g \in [0, 1]$ represents the degree of product differentiation, where if $g = 0$ firms sell completely differentiated products (and, hence, their demands are independent of each other), i.e., $p_i = 1 - q_i$ for every firm i; while if $g = 1$, firms sell homogeneous products, yielding an inverse demand function $p_i = 1 - q_i - q_j = 1 - Q$, where $Q = q_i + q_j$.

(a) a. Show that consumer surplus in this market is given by $CS = \frac{1}{2}(q_B^2 + 2gq_Bq_G + q_G^2)$. Interpret.

- For firm i, at the equilibrium point, $p_i = 1 - q_i - gq_j$

$$\begin{aligned} CS_i &= \frac{1}{2}(1 - p_i) q_i \\ &= \frac{1}{2}[1 - (1 - q_i - gq_j)] q_i \\ &= \frac{1}{2}(q_i + gq_j) q_i \end{aligned}$$

Hence,

$$\begin{aligned} CS &= CS_B + CS_G \\ &= \frac{1}{2}(q_B + gq_G) q_B + \frac{1}{2}(q_G + gq_B) q_G \\ &= \frac{1}{2}(q_B^2 + 2gq_Bq_G + q_G^2) \end{aligned}$$

Q.E.D.

b. *No regulation.* Find equilibrium output levels when firms do not face emission fees. Evaluate these output levels in the extreme cases of $g = 0$ and $g = 1$. Interpret.

- To maximize its profit, the optimization function for firm i will be

$$\begin{aligned}\max_{q_i} p_i(q_i, q_j)\, q_i - C_i q_i &= \max_{q_i} (1 - q_i - g q_j)\, q_i - C_i q_i \\ &= \max_{q_i} (1 - g q_j - C_i)\, q_i - q_i^2\end{aligned}$$

  F.O.C.

$$1 - C_i - g q_j - 2q_i = 0$$

  Hence,

$$\begin{aligned}q_i^\star &= \frac{1 - C_i - g q_j^\star}{2} \\ &= \frac{1 - C_i - g\frac{1 - C_i - g q_i^\star}{2}}{2} \\ &= \frac{1 - C_i}{2 + g}\end{aligned}$$

  If $g = 0$, $q_i^\star = \frac{1-C_i}{2}$; if $g = 1$, $q_i^\star = \frac{1-C_i}{3}$, which means that the firm will produce more if it sells more differentiated products.

c. *Regulation.* Find equilibrium output levels when firms face any emission fee t. Evaluate these output levels in the extreme cases of $g = 0$ and $g = 1$. Interpret.

- Here, the optimization function for firm i with tax will be

$$\max_{q_i} p_i(q_i, q_j)\, q_i - C_i q_i - t_i q_i$$

  By the similar logic as part b, we can obtain

$$q_i^\star = \frac{1 - C_i - t_i}{2 + g}$$

  If $g = 0$, $q_i^\star = \frac{1-C_i-t_i}{2}$; if $g = 1$, $q_i^\star = \frac{1-C_i-t_i}{3}$, which means that the firm will produce more if it sells more differentiated products, but it will produce less compared with the cases if no tax is imposed.

d. Identify the socially optimal output level for the brown firm, q_B^{SO}, and for the green firm, q_G^{SO}.

- The social planner has the right to decide the optimal q_B and q_G to maximize the social welfare function.

$$\max_{q_B,q_G} \frac{1}{2}\left(q_B^2+gq_Bq_G+q_G^2\right)+\left(1-q_B-gq_G\right)q_B-C_Bq_B+\left(1-q_G-gq_B\right)q_G$$
$$-C_Gq_G-d_Bq_B^2-d_Gq_G^2$$

F.O.C. w.r.t. q_B

$$q_B+gq_G+1-2q_B-gq_G-C_B-gq_G-2d_Bq_B=0$$

Further simplify,

$$\left(1+2d_B\right)q_B=1-gq_G-C_B$$

Thus,

$$\begin{aligned} q_B &= \frac{1-gq_G-C_B}{1+2d_B} \\ &= \frac{1-g\frac{1-gq_B-C_G}{1+2d_G}-C_B}{1+2d_B} \end{aligned}$$

To solve q_B,

$$\left(1+2d_B\right)q_B=1-g\frac{1-gq_B-C_G}{1+2d_G}-C_B$$

$$\left(1+2d_B\right)\left(1+2d_G\right)q_B=\left(1+2d_G\right)-g\left(1-gq_B-C_G\right)-C_B\left(1+2d_G\right)$$

$$\left[\left(1+2d_B\right)\left(1+2d_G\right)-g^2\right]q_B=\left(1+2d_G\right)\left(1-C_B\right)-g\left(1-C_G\right)$$

Hence,

$$q_B^{SO}=\frac{\left(1+2d_G\right)\left(1-C_B\right)-g\left(1-C_G\right)}{\left(1+2d_B\right)\left(1+2d_G\right)-g^2}$$

By symmetry,

$$q_G^{SO}=\frac{\left(1+2d_B\right)\left(1-C_G\right)-g\left(1-C_B\right)}{\left(1+2d_G\right)\left(1+2d_B\right)-g^2}$$

e. Find the socially optimal fees (t_B, t_G) that induce firms to produce the socially optimal output levels found in part (d).

- The social optimal tax will induce the firm to produce the same output as the social optimal output. That is to say, we need to set $q_i^\star=q_i^{SO}$. For firm B, plugging the equation of $q_B^\star$ and q_B^{SO}, we get

$$\frac{1-C_B-t_B}{2+g}=\frac{\left(1+2d_G\right)\left(1-C_B\right)-g\left(1-C_G\right)}{\left(1+2d_B\right)\left(1+2d_G\right)-g^2}$$

Hence,

$$t_B = 1 - C_B - \frac{(2+g)\left[(1+2d_G)(1-C_B) - g(1-C_G)\right]}{(1+2d_B)(1+2d_G) - g^2}$$

By symmetry, for firm G,

$$t_G = 1 - C_G - \frac{(2+g)\left[(1+2d_B)(1-C_G) - g(1-C_B)\right]}{(1+2d_G)(1+2d_B) - g^2}$$

2. Consider two consumers with utility functions over two goods, x_1 and x_2, given by

$$\begin{aligned} u_A &= \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) \quad \text{for consumer } A\text{, and} \\ u_B &= \log(x_1^B) + x_2^B - \frac{1}{2}\log(x_1^A) \quad \text{for consumer } B. \end{aligned}$$

where the consumption of good 1 by individual $i = \{A, B\}$ creates a negative externality on individual $j \neq i$ (see the third term, which enters negatively on each individual's utility function). For simplicity, consider that both individuals have the same wealth, m, and that the price for both goods is 1.

(a) *Unregulated equilibrium.* Set up consumer A's utility maximization problem, and determine his demand for goods 1 and 2, as x_1^A and x_2^A. Then operate similarly to find consumer B's demand for good 1 and 2, as x_1^B and x_2^B.

- Consumer A chooses x_1^A and x_2^A to solve

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to } \; x_1^A + x_2^A = M$$

The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M - x_1^A - x_2^A),$$

which yields first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - x_1^A - x_2^A = 0$$

Solving for x_1^A, we obtain $\frac{1}{x_1^A} = 1$, i.e., $x_1^A = 1$, which implies $M - 1 - x_2^A = 0$, or $x_2^A = M - 1$. Hence, consumer A's optimal consumption is

$$x_1^A = 1 \text{ and } x_2^A = M - 1$$

A similar argument applies to consumer B,

$$x_1^B = 1 \text{ and } x_2^B = M - 1$$

(b) *Social optimum.* Calculate the socially optimal amounts of x_1^A, x_2^A, x_1^B and x_2^B, considering that the social planner maximizes a utilitarian social welfare function, namely, $W = U_A + U_B$.

- The socially optimal consumption in this case solves

$$\max_{(x_1^A, x_2^A)} U^A + U^B \quad \text{subject to } x_1^A + x_2^A = M \text{ and } x_1^B + x_2^B = M$$

The Lagrangian for this social planner's problem is

$$\mathcal{L} = \frac{1}{2}\log(x_1^A) + \frac{1}{2}\log(x_1^B) + x_2^A + x_2^B + \lambda^A(M - x_1^A - x_2^A) + \lambda^B(M - x_1^B - x_2^B)$$

Taking first-order conditions, we find the socially optimal consumption profile:

$$x_1^A = \frac{1}{2} \text{ and } x_2^A = M - \frac{1}{2}$$

$$x_1^B = \frac{1}{2} \text{ and } x_2^B = M - \frac{1}{2}$$

Intuitively, the social planner recommends a lower consumption of good 1 (the good that generates the negative externality), and an increase in the consumption of good 2, for both individuals.

(c) *Restoring efficiency.* Show that the social optimum you found in part (b) can be induced by a tax on good 1 (so the after-tax price becomes $1+t$) with the revenue returned equally to both consumers in a lump-sum transfer.

- With tax t^A placed on good 1 and with lump-sum transfer T^A, consumer A solves

$$\max_{(x_1^A, x_2^A)} \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B)$$

$$\text{subject to } (1+t^A)x_1^A + x_2^A = M + T^A$$

where note that the price of good 1 increased from 1 to $(1+t^A)$, but this consumer also sees his wealth increase by the lump sum T^A. The Lagrangian for this optimization problem is

$$\mathcal{L} = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B) + \lambda^A(M + T^A - (1+t^A)x_1^A - x_2^A)$$

Taking first-order conditions, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1^A} = \frac{1}{x_1^A} - \lambda^A(1+t^A) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^A} = 1 - \lambda^A = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M + T^A - (1+t^A)x_1^A - x_2^A = 0$$

Simultaneously solving for x_1^A and x_2^A, we find that consumer A's consumption bundles after introducing the tax become

$$x_1^A = \frac{1}{1+t^A} \quad \text{and} \quad x_2^A = M + T^A - 1$$

Similarly we find the optimal consumption of consumer B who pays tax t^Bon good 1 and receives T^Bas a lump-sum transfer:

$$x_1^B = \frac{1}{1+t^B} \quad \text{and} \quad x_2^B = M + T^B - 1$$

- *Comparison.* Comparing the optimal consumption levels found in part (b) with the equilibrium outcomes found in part (c), the tax imposed on any individual $i = A, B$ must hence satisfy

$$\frac{1}{2} = \frac{1}{1+t^i},$$

which would guarantee that equilibrium and socially optimal amounts coincide. Solving for the tax t^i yields $t^i = \$1$. Hence, by setting a tax of $t^i = \$1$ on the consumption of good 1, and returning the tax revenue to this individual in a lump-sum transfer, efficiency is restored, yielding a consumption

$$x_1^i = \frac{1}{1+1} = \frac{1}{2} \quad \text{of good 1,}$$

and

$$\begin{aligned} x_2^i &= M + T^i - 1 \\ &= M + \frac{1}{2} - 1 = M - \frac{1}{2} \text{ of good 2,} \end{aligned}$$

as described in the socially optimal amounts found in part (b).