EconS 526- Homework #2 (Due on November 14th, 2018)

Answer Key

Question #1

Show that if A and B are square matrices of the same order, then $A^2 - B^2$ is not in general equal to $(A+B)(A-B)$

Solution

$$(A+B)(A-B) = A^2 - AB + BA - B^2$$

The multiplication of matrices is not commutative (i.e. $AB \neq BA$). In that case, $-AB + BA \neq 0$ in the above equation. Thus,

$$A^2 - B^2 \neq (A+B)(A-B)$$

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 7 \\ -4 & 13 \end{bmatrix}$$

Therefore, $AB \neq BA$ **Verified.**

Question #2

Find all 2×2 matrices A such that A^2 is the matrix obtained from A by squaring each entry.

Solution

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^2 = \begin{bmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{bmatrix}$

To prove this property, the following are required

When $b = 0$ and $c = 0$

$$A^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

When $b = 0$ and c=1

$$A^2 = \begin{bmatrix} a^2 & 0 \\ (a+b)^2 & b^2 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & 0 \\ (a+b) & b \end{bmatrix}$$

When $b = 1$ and $c = 1$

$$A^2 = \begin{bmatrix} a^2 & (a+b)^2 \\ 0 & b^2 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & (a+b) \\ 0 & b \end{bmatrix}$$

Hence, the required matrices that satisfied this property are those of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} a & 0 \\ (a+b) & b \end{bmatrix}, \text{ or } \begin{bmatrix} a & (a+b) \\ 0 & b \end{bmatrix}$$

If $a=2$ and $b=1$

$$A=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} (2)^2 & 0 \\ 0 & (1)^2 \end{bmatrix}$$

$$A=\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$A^2=\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}=\begin{bmatrix} 4 & 0 \\ 9 & 1 \end{bmatrix}=\begin{bmatrix} (2)^2 & 0 \\ (3)^2 & (1)^2 \end{bmatrix}$$

$$A=\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^2=\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 4 & 9 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} (2)^2 & (3)^2 \\ 0 & (1)^2 \end{bmatrix}$$

Verified

Question #3

Calculate AB when

$$A=\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \qquad B=\begin{bmatrix} 3 & -1 & 6 \\ 0 & 2 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

i. What general result about upper triangular matrices does your result suggest?
ii. What is the particular result for lower triangular matrices?

Solution

$$AB=\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}\begin{bmatrix} 3 & -1 & 6 \\ 0 & 2 & 1 \\ 0 & 0 & -5 \end{bmatrix}=\begin{bmatrix} 3a & -a+2b & 6a+b-5c \\ 0 & 2d & d-5e \\ 0 & 0 & -5f \end{bmatrix}$$

i. The product of two upper (lower) triangular matrices of the same order is upper (lower) triangular.

ii. $$\begin{bmatrix} * & -a+2b & 6a+b-5c \\ 0 & * & d-5e \\ 0 & 0 & * \end{bmatrix}$$

Question #4

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and let y be a given 3-vector with components y_1, y_2, y_3.

Solve each of the following systems of equations:

i. $Ax = y$

ii. $Bx = y$

iii. $Cx = y$

iv. What feature of the solution procedure distinguishes (iii) from (i) and (ii)?

Solution

i. $Ax = y$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Writing out the system of equations, we have

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= y_1 \\ -x_2 + x_3 &= y_2 \\ 2x_3 &= y_3 \end{aligned}$$

Using substitution method, $x_3 = \dfrac{y_3}{2}, x_2 = \dfrac{y_3}{2} - y_2, x_1 = \dfrac{y_1 - y_2 - y_3}{2}$

ii. $Bx = y$

$$\begin{bmatrix} 4 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Writing out the system of equations, we have

$$\begin{aligned} 4x_1 &= y_1 \\ 3x_1 - 2x_2 &= y_2 \\ x_1 + x_2 - x_3 &= y_3 \end{aligned}$$

Using substitution method, $x_1 = \dfrac{y_1}{4}, x_2 = \dfrac{3y_1 - 4y_2}{8}, x_3 = \dfrac{5y_1 - 4y_2 - 8y_3}{8}$

iii. $Cx = y$

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Writing out the system of equations, we have

$$-3x_1 = y_1$$
$$2x_2 = y_2$$
$$x_3 = y_3$$

Using substitution method, $x_1 = -\frac{y_1}{3}, x_2 = \frac{y_2}{2}, x_3 = y_1$

iv. In (iii), the **diagonal entries** of matrix C constituted the solution to the system of equations. On the other hand, **back-substitution** and **forward substitution** established the solution procedures to the system of equations in (i) and (ii) respectively.

Question #5

Find the determinants of the following matrices, and show that

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$A = \begin{bmatrix} 1 & 5 & 6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ -1 & 0 & -9 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 0 & 4 \\ -1 & 2 & 8 & 5 \\ -1 & -1 & -2 & 3 \end{bmatrix}$$

Solution:

(a) $|A| = -9$

(b) $|B| = 0$

(c) $|C| = 60$

For the second part of the problem, please assume matrix A, B and C all have conformable dimensions. For example, you can assume A, B and C are all 2×2 matrices. Also assume A, B and C have non-zero determinants so all the three matrix have inverse matrices.

According to Theorem 8.10 (c) from S&B, we know that $(AB)^{-1} = B^{-1}A^{-1}$

In this problem, $(ABC)^{-1} = [(AB)C]^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$