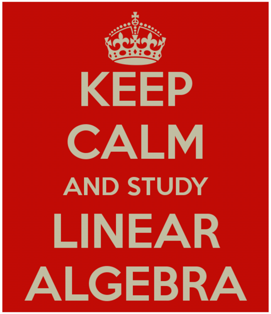
Determinants

Chapter 9 - S&B



Overview

- The most important matrices in economic models are square matrices (number of unknowns equal number of equations)
- The most important matrices are the nonsingular
- $Ax = b$ has a unique solution
- If the determinant of a square matrix is not zero, then the matrix is nonsingular

The Determinant

- Matrix A is nonsingular if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Therefore,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- More generally

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\det(a_{22}) - a_{12}\det(a_{21})$$

The Determinant

Definition

Let A be an $n \times m$ matrix. Let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. Then the scalar

$$M_{ij} \equiv \det A_{ij}$$

is called the (i,j)th **minor** of A and the scalar

$$C_{ij} \equiv (-1)^{i+j} M_{ij}$$

is called the (i,j)th **cofactor** of A, where $M_{ij} = C_{ij}$ if $(i+j)$ is even and $M_{ij} = -C_{ij}$ if $(i+j)$ is odd. Then

$$\det A = a_{11} M_{11} - a_{12} M_{12} = a_{11} C_{11} + a_{12} C_{12}$$

The Determinant

Definition

The determinant of a 3×3 matrix is given by

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$+ a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The Determinant

Example

Let's consider

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$$

The Determinant

Definition

The determinant of a $n \times n$ matrix A is given by

$$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n} \\ &= a_{11}M_{11} - a_{12}M_{12} + \ldots + (-1)^{n+1}a_{1n}M_{1n} \end{aligned}$$

Computing the Determinant

- Our definition of the determinant of a matrix involves expanding along its 1st row.
- THERE IS NOTHING SPECIAL ABOUT THE 1st ROW!
- We can use any row or column to compute the determinant of a 3×3 matrix (use the second column)

$$
\begin{aligned}
\det A &= -a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{22} \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \\
&\quad - a_{32} \det \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}
\end{aligned}
$$

Computing the Determinant

Theorem

The determinant of a lower-triangular, upper-triangular, or diagonal matrix is simply the product of its diagonal entries.

Examples

$$A = \begin{pmatrix} 1 & 0 \\ 8 & 4 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 1 & 5 \end{pmatrix}$$

Main Property of the Determinant

- The determinant determines whether or not a square matrix is nonsingular

Example

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix} \qquad \begin{matrix} 2 & 4 & 0 \\ 4 & 6 & 3 \end{matrix}$$

Uses of Determinants

- The Determinant tells whether or not A^{-1} exists and whether or not $Ax = b$ has a unique solution.
- We can use the determinant to derive a formula foe A^{-1} and a formula for the solution x of $Ax = b$.
- **Theorem 9.4**. Let A be a nonsingular matrix the:
  - $A^{-1} = \frac{1}{\det A} \cdot adjA$, and
  - (Cramer's rule) the unique solution $x = (x_1, ..., x_n)$ of the $n \times n$ system $Ax = b$ is $x_i = \frac{\det B_i}{\det A}$, for $i = 1, ..., n$, where B_i is the matrix A with the right-hand side b replacing the ith column of A.

Uses of Determinants

Example

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$