Recitation #6 (09/28/2018) - Production Theory

1. Consider the following profit function that has been obtained from a technology that uses a single input, z:

$$\pi(p, w) = p^2 w^\alpha$$

where p is the output price, w is the input price and α is a parameter value.

(a) Check if the profit function satisfies homogeneity of degree one jointly in both p and w. In particular, determine for which values of α this property is satisfied.

- The profit function is homogeneous of degree one if

$$\pi(\theta p, \theta w) = \theta \pi(p, w)$$

In this case we have that the left-hand term becomes

$$\pi(\theta p, \theta w) = (\theta p)^2 (\theta w)^\alpha = \theta^{2+\alpha} p^2 w^\alpha \tag{3}$$

and, on the other hand, the right-hand term is

$$\theta \pi(p, w) = \theta p^2 w^\alpha \tag{4}$$

since, by homogeneity of degree one, expressions (3) and (4) must coincide. Then,

$$\theta^{2+\alpha} p^2 w^\alpha = \theta p^2 w^\alpha$$

which implies that $2 + \alpha = 1$. That is, we need $\alpha = -1$. As a consequence, the profit function that we obtain is

$$\pi(p, w) = \frac{p^2}{w}$$

(b) Assuming the value of α for which the profit function satisfies homogeneity of degree one, check if the profit function $\pi(p, w)$ satisfies the following properties: (1) non-decreasing in output price p, (2) non-increasing in input prices w, and (3) convex in prices p and w.

- *Non-decreasing in the output price, p:* Increasing output prices yields a weakly higher profit level since

$$\frac{\partial \pi(p, w)}{\partial p} = \frac{2p}{w} \geq 0$$

- *Non-increasing in the factor prices, w:* Increasing all input prices weakly reduces profits since
$$\frac{\partial \pi(p,w)}{\partial w} = -\frac{p^2}{w^2} \leq 0$$
- *Convex in prices* (factor prices and output prices):
$$\begin{vmatrix} \frac{\partial^2 \pi(p,w)}{\partial p^2} & \frac{\partial^2 \pi(p,w)}{\partial p \partial w} \\ \frac{\partial^2 \pi(p,w)}{\partial w \partial p} & \frac{\partial^2 \pi(p,w)}{\partial w^2} \end{vmatrix} = \begin{vmatrix} \frac{2}{w} & -\frac{2p}{w^2} \\ -\frac{2p}{w^2} & \frac{2p^2}{w^3} \end{vmatrix}$$

  In particular, the Hessian is a positive semi-definite matrix, since
$$\frac{2}{w}\frac{2p^2}{w^3} - \left(-\frac{2p}{w^2}\right)\left(-\frac{2p}{w^2}\right) = \frac{4p^2}{w^4} - \frac{4p^2}{w^4} = 0$$
  implying that the profit function $\pi(p,w)$ is convex.

(c) Calculate the supply function of the firm, $q(p,w)$, and its demand for inputs, $z(p,w)$.

- Using Hotelling's Lemma we can find the supply function, by differentiating the profit function with respect to p, as follows
$$q(p,\, w) = \frac{\partial \pi(p,w)}{\partial p} = \frac{2p}{w}$$
  and the conditional factor demand correspondence can also be found by differentiating the profit function with respect to w, as follows
$$z(p,\, w) = -\frac{\partial \pi(p,w)}{\partial w} = \frac{p^2}{w^2}$$
- Note that both the supply function, $q(p,\, w)$, and the input demand function, $z(p,\, w)$, are increasing in output prices p (more attractive sales) but decreasing in input prices w (i.e., more costly resources).

2. The profit function, $\pi(p)$, is defined as
$$\pi(p) = \max\{p \cdot y \mid y \in Y\}$$
or alternatively, $\pi(p) \geqslant p \cdot y$ for every feasible production plan $y \in Y$.

(a) Show that the profit function $\pi(p)$ is convex in prices.

- We need to show that, for any two price vectors p,$p' \in \mathbb{R}^L_{++}$,

$$\pi(\alpha p + (1-\alpha)p') \leq \alpha\pi(p) + (1-\alpha)\pi(p') \quad \text{where } \alpha \in [0,1]$$

Proof. From the definition of the profit function we know that $\pi(p)$ is the maximal of $p \cdot y$, so we have

$$\begin{aligned} \pi(p) &\geq p \cdot y, \text{ for any } y \in Y \text{ and } p >> 0, \text{ and} \\ \pi(p') &\geq p' \cdot y, \text{ for any } y \in Y \text{ and } p' >> 0 \end{aligned}$$

We now multiply all prices by a common factor $\alpha \in [0,1]$, thus obtaining

$$\begin{aligned} \alpha\pi(p) &\geq \alpha p \cdot y, \text{ for any } y \in Y \text{ and } p >> 0, \text{ and} \\ (1-\alpha)\pi(p') &\geq (1-\alpha)p' \cdot y, \text{ for any } y \in Y \text{ and } p' >> 0 \end{aligned}$$

Adding up the two previous inequalities, yields

$$\alpha\pi(p) + (1-\alpha)\pi(p') \geq \alpha p \cdot y + (1-\alpha)p' \cdot y = [\alpha p + (1-\alpha)p'] \cdot y$$

where the right-hand side, $[\alpha p + (1-\alpha)p'] \cdot y$, coincides with the profit function at price level $\alpha p + (1-\alpha)p'$, i.e., $\pi(\alpha p + (1-\alpha)p')$. Hence, the above inequality can be more compactly expressed as

$$\alpha\pi(p) + (1-\alpha)\pi(p') \geq \pi(\alpha p + (1-\alpha)p')$$

Therefore, the profit function is convex in prices.

(b) Prove that Hotelling's lemma holds for this profit function.

- Hotelling's lemma states that if the output function evaluated at price vector p, $y(\overline{p})$, consists of a single output level, then the profit function $\pi(p)$ is differentiable at the price level $\overline{p}$; and moreover such derivative is $\nabla_p.\pi(\overline{p}) = y(\overline{p})$.
- Let us first express the profit function as the support function that, for every price vector p, chooses the infimum of $p \cdot (-y)$, i.e., instead of choosing the max of $p \cdot y$. We, hence, redefine it as the inf of $p \cdot (-y)$, as follows

$$\pi(p) = \inf\{p \cdot (-y) \,|y \in Y\}$$

- In order to emphasize the similarities with the Duality Theorem, we reproduce this theorem next: Let K be a non-empty and closed set, and let $\mu_K(p)$ be its support function, that is $\mu_K(\overline{p}) = \inf\{p \cdot x | x \in X\}$. Then, there exists a unique element $\overline{x}$ in set K such that $\mu_K(\overline{p}) = \overline{p} \cdot \overline{x}$ if and only if $\mu_K(\overline{p})$ is differentiable at $\overline{p}$. Moreover, $\nabla_p \mu_K(\overline{p}) = \overline{x}$.

- Therefore, given that we have noticed that the profit function can be expressed as a support function, we can rewrite Hotelling's lemma as a direct application of the Duality Theorem (just changing labels!): Let $-Y$ be a non-empty and closed (production) set, and let $\mu_{-Y}(p)$ be its support function. Then, there exists a unique production function $y(\overline{p})$ in the set $-Y$ such that $\pi(\overline{p}) = \overline{p} \cdot y(\overline{p})$ if and only if $\pi(\overline{p})$ is differentiable at $\overline{p}$. Moreover, this derivative is $\nabla_p \pi(\overline{p}) = y(\overline{p})$.

(c) Show that, if the output function $y(p)$ is differentiable at $\overline{p}$, then $D_p y(\overline{p})$ is a symmetric and positive semidefinite matrix.

- *Symmetry*: This comes from two sources. First, own-substitution effects are nonnegative, i.e., $\frac{\partial y_l(p)}{\partial p_l} \geq 0$ for all l. Second, using Young's Theorem, cross-substitution effects are symmetric, i.e., $\frac{\partial y_l(p)}{\partial p_k} = \frac{\partial y_k(p)}{\partial p_l}$ for all $l \neq k$.
- *Positive semidefinite matrix*: It is the mathematical expression for the law of supply: quantities respond in the same direction as price changes. (See exercise #6 in this chapter for more details.) That is given that there are no wealth effects,