Recitation #8 - (10/19/2018)

1. Suppose there are two periods, 'today' (i.e. period 1) and 'tomorrow' (i.e. period 2), and a single consumption good. An individual called Andreu (not necessarily the same person as the first author in Mas-Colell, Whinston and Green's *Microeconomic Theory*) has preferences over two-period consumption streams that are additively separable. In particular assume his preferences admit a representation of the form:

$$U(x_1, x_2) = u(x_1) + \frac{1}{1+r}u(x_2),$$

where $r > 0$ is the interest rate at which the agent can borrow or lend money, and

$$u(x_t) = -\frac{(x_t - 2)^2}{2}$$

for every time period $t = \{1, 2\}$.

(a) If his income today is $y_1 = 1$ and he knows that his income tomorrow will also be $y_2 = 1$, and solve for her optimal consumption in each period and calculate the level of discounted lifetime utility he achieves.

- Andreu's utility maximization problem is to optimally choose x_1 and x_2 to solve

$$\begin{aligned} \max_{x_1,x_2} U(x_1, x_2) &= u(x_1) + \beta u(x_2) \\ \text{subject to } x_1 &= y_1 - s, \text{ and} \\ x_2 &= y_2 + (1+r)s \end{aligned}$$

where s denotes his savings. Simplifying, we can more compactly express this maximization problem as an unconstrained problem where Andreu only needs to choose a single choice variable, s, as follows

$$\max_s \; -\frac{(y_1 - 2 - s)^2}{2} - \frac{1}{1+r}\frac{(y_2 - 2 + (1+r)s)^2}{2}$$

- Taking first order conditions with respect to s, we obtain

$$(y_1 - 2 - s) - (y_2 - 2 + (1+r)s) = 0$$

Rearranging and solving for s, we find optimal savings of

$$s = \frac{y_1 - y_2}{2+r}.$$

- Hence, when $y_1 = 1$ and $y_2 = 1$, Andreu's savings are zero, i.e., $s = \frac{1-1}{2+r} = 0$, and his utility level becomes

$$U(1,1) = -\frac{1}{2} - \frac{1}{2(1+r)} = -\frac{(3+r)}{2(1+r)}$$

Suppose that Andreu still knows that his income today is $y_1 = 1$. He is, however, uncertain about his future payoff y_2: It could be high, $y_2^H = 3/2$ or low, $y_2^L = 1/2$, each happening with equal probability. His problem now is to maximize his discounted expected utility by choosing the initial period consumption x_1; his future consumption if his income tomorrow is high, x_2^H ; and his future consumption if income tomorrow is low, x_2^L.

b. Formulate the agent's optimization problem and solve for the optimal consumption plan and his level of discounted expected utility.

- Andreu's optimization problem now becomes selecting x_1, x_2^H and x_2^L to solve

$$\max_{x_1, x_2^H, x_2^L} \quad -\frac{(x_1-2)^2}{2} - \frac{1}{1+r}\left[\frac{1}{2}\frac{(x_2^H-2)^2}{2} + \frac{1}{2}\frac{(x_2^L-2)^2}{2}\right]$$

$$\begin{aligned} \text{subject to } x_1 &= y_1 - s, \\ x_2^H &= y_2^H + (1+r)s, \text{ and} \\ x_2^L &= y_2^L + (1+r)s. \end{aligned}$$

Rearranging, we can express this optimization problem as an unconstrained maximization problem, where now Andreu's single choice variable is his savings, s, as follows

$$\max_s \quad -\frac{(y_1-2-s)^2}{2} - \frac{1}{1+r}\left[\frac{1}{2}\frac{(y_2^H-2+(1+r)s)^2}{2} + \frac{1}{2}\frac{(y_2^L-2+(1+r)s)^2}{2}\right]$$

- Taking first order conditions with respect to s, we obtain

$$(y_1-2-s) - \frac{1}{2}\left(y_2^H - 2 + (1+r)s\right) - \frac{1}{2}\left(y_2^L - 2 + (1+r)s\right) = 0$$

rearranging,

$$y_1 - \frac{1}{2}y_2^H - \frac{1}{2}y_2^L = s(2+r),$$

and solving for s yields

$$s = \frac{y_1 - E[y_2]}{2+r}$$

where $E[y_2] = \frac{1}{2}y_2^H + \frac{1}{2}y_2^L = \frac{1}{2}\frac{3}{2} + \frac{1}{2}\frac{1}{2} = 1$ denotes the expected second-period income.

c. Compare your answers to parts (a) and (b).

- In expectation, Andreu's savings when he is certain about his future income being $y_2 = 1$ (in part a of the exercise) and when he is uncertain (but his expected income is $E[y_2] = 1$) coincide. Intuitively, Andreu's savings take into account that his future income is, in expectation, 1.

2. Consider a cumulative distribution function $F(x)$ which first-order stochastically dominates $G(x)$.

(a) Show that the mean of x under $G(x)$, $\int x \, dG(x)$, cannot exceed that under $F(x)$, $\int x \, dF(x)$, i.e.,

$$\int xdF(x) \geqslant \int xdG(x)$$

- We know that distribution function $F(x)$ first-order stochastically dominates $G(x)$ if

$$\int u(x) \, dF(x) \geqslant \int u(x) \, dG(x)$$

Using the fact that the utility function is weakly increasing, we have

$$\int xdF(x) \geqslant \int xdG(x)$$

(b) Provide now an example where $\int xdF(x) \geqslant \int xdG(x)$ is satisfied, but $F(x)$ does not first order stochastically dominates $G(x)$.

- Consider the two lotteries depicted in figure 1. The first one, $F(x)$, assigns 1/2 to monetary outcome \$2 and 1/2 to \$3. The second lottery, $G(x)$, evenly splits the probability weight that lottery $F(x)$ assigns to \$2 between \$1 and \$2 (each occurring with a probability of 1/4), and the probability weight of

\$3 is also equally divided between \$3 and \$4.

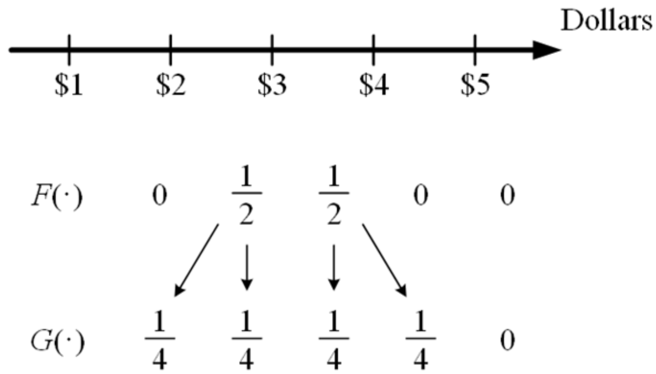

Figure 1. Lotteries $F(x)$ and.$G(x)$.

- Thus, both lotteries $F(x)$ and $G(x)$ have the same expected value, $\frac{5}{2}$. However, neither $F(x)$ FOSD $G(x)$, nor $G(x)$ FOSD $F(x)$. In particular, depicting both probability distributions (see figure 2), one can easily observe that lottery $G(x)$ lies weakly above $F(x)$ for outcomes $x \leq \$3$, but lies below for monetary outcomes beyond that threshold.

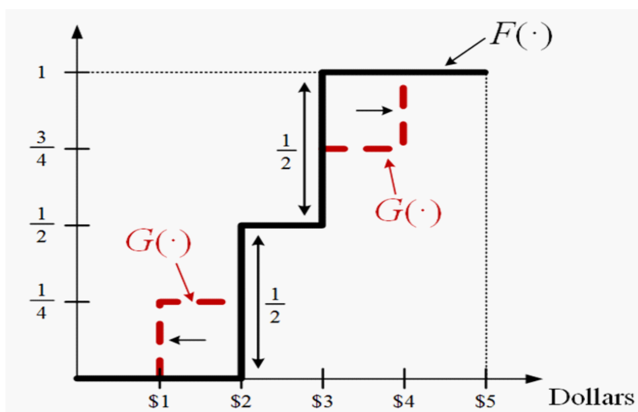

Figure 2. Lotteries $F(x)$ and.$G(x)$.