Midterm #1 EconS 527

Monday, October 8th, 2018

NAME: ____________________

Instructions. *Show all your work clearly and make sure you justify all your answers.*

1. **Question 1 [15 Points].** Consider a lottery over three possible monetary outcomes:

1st Prize	2nd Prize	3rd Prize
\$2.5 million	\$500,000	\$0

The first choice set is $L_1 = (0, 1, 0)$ and $L_2 = (\frac{10}{100}, \frac{89}{100}, \frac{1}{100})$ and the second choice set is $L_1 = (0, \frac{11}{100}, \frac{89}{100})$ and $L_2 = (\frac{10}{100}, 0, \frac{90}{100})$. Show that the IA is violated in this setting.

Solution

Check Lecture Notes

2. **Question 2 [20 Points].** Let us consider the case of two goods, $L = 2$. Then, an individual prefers a bundle $x = (x_1, x_2)$ to another bundle $y = (y_1, y_2)$ if and only if it contains more units of both goods than bundle y, i.e. $x_1 \geq y_1$ and $x_2 \geq y_2$. Informally, *he prefers more of everything.* Check if this preference relation satisfies: (1) completeness, (2) transitivity and (3) strong monotonicity. Assume a bundle equal to $(4, 2)$ and use a graph to support your answer.

Solution

Completeness: First, the upper contour set (UCS) describes the region of bundles that the consumer prefers to bundle $(4, 2)$. The UCS of bundle $(4, 2)$ thus contains those bundles (x_1, x_2) with weakly more than two units of good 1 and weakly more than one unit of good 1, that is

$$UCS(4,2) = \{(x_1, x_2) \succsim (4,2) \Leftrightarrow x_1 \geq 4 \textit{ and } x_2 \geq 2\}$$

as depicted in the shaded region of figure 1 labeled UCS, which lies to the northeast $(4, 2)$.

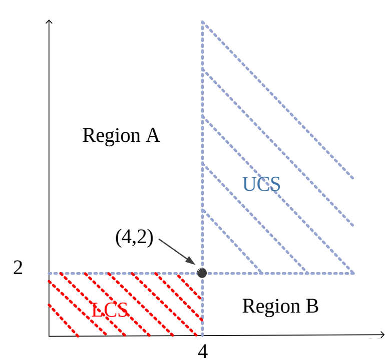

Figure 1

In contrast, the bundles in the lower contour set of bundle $(4, 2)$, whereby the consumer is better off with bundle $(4, 2)$, contain fewer units of both goods, that is,

$$LCS(4,2) = \{(4,2) \succsim (x_1, x_2) \Leftrightarrow x_1 \leq 4 \textit{ and } x_2 \leq 2\}$$

The lower contour set (LCS) is therefore illustrated by those points to the southwest of $(4, 2)$; as depicted in figure 1.

Finally, the indifference set, IND $(4, 2)$, which comprises those bundles (x_1, x_2) for which the consumer is indifferent between $(4, 2)$ and (x_1, x_2), is empty, since there are no regions for which the upper contour set and the lower contour set overlap.

Hence, it is not complete, since completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both).

Transitivity: It is transitive, since transitivity requires that, for any three bundles x, y and z, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$. Now $x \succsim$y and $y \succsim z$ means that $x \geq y$ and $y \geq z$.

In vector notation, this means that bundle x is weakly larger than y in every component. Similarly, bundle y is weakly larger than bundle z in every component. Hence,

$x \geq y$ and $y \geq z$, which implies that $x_1 \geq y_1$ and $y_1 \geq z_1$ for every good l.

Strong monotonicity: It is strongly monotone. The property of strong monotonicity requires that if we increase one of the goods in a given bundle, then the newly created bundle must be strictly preferred to the original bundle. More compactly, for a given bundle y, if bundle x satisfies

$$x \geq y \text{ and } x \neq y \text{ then } x \succ y$$

In words, bundle x being weakly larger than y in all components (but being strictly larger in at least one component, since they cannot completely coincide) implies that bundle x is weakly preferred to y, and it can never be that bundle y is weakly preferred to bundle x. We can therefore conclude that x is strictly preferred to y, $x \succ y$, as required.

3. **Question 3 [20 Points].** Consider an individual with utility function

$$u(x_1, x_2) = \ln x_1 + x_2,$$

where x_1 and x_2 denote the amounts consumed of non-organic and organic goods, respectively. The prices of these goods are $p_1 > 0$ and $p_2 > 0$, respectively; and this individual's wealth is $w > 0$.

(a) Find this consumer's uncompensated demand for every good $x_i(p, w)$, where $i = \{1, 2\}$. [For compactness, we use p to denote the price vector $p \equiv (p_1, p_2)$.] Under which conditions the consumer demands positive amounts of both goods? Interpret your results.

- The tangency condition for this consumer, $MRS = \frac{p_1}{p_2}$, becomes

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{1}{x_1} = \frac{p_1}{p_2}$$

which simplifies to $p_1 x_1 = p_2$. Solving for x_1, we obtain the Walrasian demand for the non-organic good,

$$x_1(p, w) = \frac{p_2}{p_1}.$$

Substituting this Walrasian demand into the budget constraint $p_1 x_1 + p_2 x_2 = w$ yields

$$p_1 \underbrace{\frac{p_2}{p_1}}_{x_1} + p_2 x_2 = w.$$

Solving for x_2, we find the Walrasian demand for good 2 (organic good),

$$x_2(p, w) = \frac{w}{p_2} - 1$$

which is positive as long as $\frac{w}{p_2} > 1$, or if wealth w is sufficiently high, $w > p_2$. In this context, the consumer buys positive units of both organic and non-organic goods. Otherwise, the consumer only purchases a positive amount of the non-organic good $x_1(p, w) > 0$ but a zero amount of the organic good, $x_2(p, w) = 0$. Intuitively, this occurs when her income is relatively low.

- This result is due to the quasilinear utility function, leading the consumer to purchase strictly positive units of the good entering non-linearly (good 1) under all parameter values, but zero units of the good entering linearly (good 2) under relatively general parameter conditions.

(b) Find the indirect utility function, $v(p, w)$.

- Substituting the above Walrasian demands into the utility function gives the indirect utility function

$$\begin{aligned} v(p,w) &= \ln x_1(p,w) + x_2(p,w) \\ &= \ln\left(\frac{p_2}{p_1}\right) + \left(\frac{w}{p_2} - 1\right) \end{aligned}$$

(c) Find this consumer's expenditure function, $e(p,v)$, and her compensated demand for every good $h_i(p,w)$, where $i = \{1,2\}$.

- *Expenditure function.* Solving for wealth w in the indirect utility function we found in part (a), $v(p,w)$, yields the expenditure function. Setting $v = v(p,w)$ and rearranging the indirect utility function, we obtain

$$v - \ln\left(\frac{p_2}{p_1}\right) + 1 = \frac{w}{p_2}$$

and solving for w, yields the expenditure function

$$e(p,v) = p_2\left[v - \ln\left(\frac{p_2}{p_1}\right) + 1\right]$$

- *Hicksian demands.* By Shepard's lemma, $h_1(p,v) = \frac{\partial e(p,v)}{\partial p_1}$, we can find Hicksian (compensated) demands by differentiating our above expenditure function with respect to the price of each good, as follows,

$$\begin{aligned} h_1(p,v) &= \frac{\partial e(p,v)}{\partial p_1} = \frac{p_2}{p_1}, \text{ and} \\ h_2(p,v) &= \frac{\partial e(p,v)}{\partial p_2} = v - \ln\left(\frac{p_2}{p_1}\right) \end{aligned}$$

Alternatively, we can also find Hicksian (compensated) demands by evaluating the Walrasian (uncompensated) demands at a wealth that coincides with the expenditure function, that is, $w = e(p,v)$, yielding

$$h_1(p,v) = x_1(p,e(p,v)) = \frac{p_2}{p_1}$$

for good 1 (since its Walrasian demand is independent of income, $x_1(p,w) = \frac{p_2}{p_1}$), and

$$h_2(p,v) = x_2(p,e(p,v)) = \frac{\overbrace{p_2\left[v - \ln\left(\frac{p_2}{p_1}\right) + 1\right]}^{w=e(p,v)}}{p_2} - 1$$

for good 2, which simplifies to

$$\begin{aligned} h_2(p,v) &= \left[v - \ln\left(\frac{p_2}{p_1}\right) + 1\right] - 1 \\ &= v - \ln\left(\frac{p_2}{p_1}\right) \end{aligned}$$

The Hicksian (compensated) demand for good 1 (organic) is independent of the utility level that the consumer targets in her expenditure minimization problem, v; but her Hicksian demand for good 2 (non-organic) is increasing in this utility level he seeks to target.

(d) Solve parts (a)-(c) of the exercise again, but considering that the consumer's utility function is now $u(x_1,x_2) = (x_1 - a_1)(x_2 - a_2)$, where parameters a_1 and a_2 are both weakly positive, $a_1, a_2 \geq 0$.

- *Finding Walrasian demand.* The tangency condition for this consumer, $MRS = \frac{p_1}{p_2}$, becomes

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{x_2 - a_2}{x_1 - a_1} = \frac{p_1}{p_2}$$

which simplifies to $p_1x_1 = p_1a_1 - p_2a_2 + p_2x_2$. Substituting this result into the budget constraint, $p_1x_1 + p_2x_2 = w$ yields

$$\underbrace{(p_1a_1 - p_2a_2 + p_2x_2)}_{p_1x_1} + p_2x_2 = w.$$

which simplifies to $p_1a_1 + p_2(a_2 - 2x_2) = w$. Solving for x_2, we obtain the Walrasian demand for good 2 (organic)

$$x_2(p, w) = \frac{w - p_1a_1 + p_2a_2}{2p_2}.$$

Inserting this result into the budget constraint, yields

$$p_1x_1 + p_2\underbrace{\left(\frac{w - p_1a_1 + p_2a_2}{2p_2}\right)}_{x_2(p,w)} = w$$

Solving for x_1, we find the Walrasian demand for good 1 (non-organic) to be

$$x_1(p, w) = \frac{w + p_1a_1 - p_2a_2}{2p_1}.$$

The Walrasian demand for good 2 (organic) is positive as long as $a_1 < \frac{w+p_2a_2}{p_1}$, whereas the Walrasian demand for good 1 (non-organic) is positive as long as $a_2 < \frac{w+p_1a_1}{p_2}$. Intuitively, the minimal amounts that the consumer needs to consume to obtain a positive utility level must be sufficiently small for her Walrasian demands to be positive.

- The Walrasian demand of every good i is increasing in the minimal amount that the consumer needs from that good a_i, but decreasing in the minimal amount that the consumer needs from the other good a_j. For instance, if the consumer does not need any positive amount of organic food but requires a large amount of non-organic food, $a_1 > 0$ but $a_2 = 0$, the above Walrasian demands collapse to

$$x_1(p, w) = \frac{w + p_1a_1}{2p_1} \quad \text{and} \quad x_2(p, w) = \frac{w - p_1a_1}{2p_2}$$

- *Indirect utility function.* Substituting the above Walrasian demands into the utility function gives the indirect utility function

$$\begin{aligned} v(p, w) &= (x_1(p, w) - a_1)(x_2(p, w) - a_2) \\ &= \left(\frac{w + p_1a_1 - p_2a_2}{2p_1} - a_1\right)\left(\frac{w - p_1a_1 + p_2a_2}{2p_2} - a_2\right) \\ &= \frac{(w - p_1a_1 - p_2a_2)^2}{4p_1p_2} \end{aligned}$$

- *Expenditure function.* Solving for wealth w in the indirect utility function we found in part (a), $v(p, w)$, yields the expenditure function. Setting $v = v(p, w)$, applying square roots on both sides, and rearranging the indirect utility function, we obtain

$$\sqrt{v} = \frac{w - p_1a_1 - p_2a_2}{2\sqrt{p_1p_2}}$$

and solving for w, yields the expenditure function

$$e(p, v) = 2\sqrt{vp_1p_2} + p_1a_1 + p_2a_2$$

- *Hicksian demands.* By Shepard's lemma, $h_1(p, v) = \frac{\partial e(p,v)}{\partial p_1}$, we can find Hicksian (compensated) demands by differentiating our above expenditure function with respect to the price of each good, as follows,

$$\begin{aligned} h_1(p, v) &= \frac{\partial e(p, v)}{\partial p_1} = a_1 + \sqrt{v\frac{p_2}{p_1}}, \text{ and} \\ h_2(p, v) &= \frac{\partial e(p, v)}{\partial p_2} = va_2 + \sqrt{v\frac{p_1}{p_2}}. \end{aligned}$$

4. **Question 4 [20 Points].** Luis has a monthly income of \$80. He spends his money sending text messages (measured in minutes) at a price p_x and on other composite good y, whose price has been normalized to one, i.e., $p_y = \$1$. His mobile phone company offers him two plans: Plan A, in which he pays no monthly fee and sends text messages for \$0.25 per minute; or Plan B, in which he pays a \$40 monthly fee and benefits from cheaper text messages at \$0.10 per minute.

   (a) Depict Luis's budget constraint under each of the two plans, with the number of texts (good x) in the horizontal axis and the composite good (good y) in the vertical axis. In addition, identify the intersection point between the two budget constraints.

   (b) If Luis mentions that Plan A is better for him, what is the set of baskets he may purchase if his behavior is consistent with the WARP?

Solution

2. (a) Depict Luis's budget constraint under each of the two plans, with the number of phone calls (good x) in the horizontal axis and the composite good (good y) in the vertical axis.

Let x denote the number of texts, and y denote spending on other goods. The expression of the budget line under Plan A, BL_A is $0.25x + y = 80$, or $y = 80 - 0.25x$, as depicted in the solid line of figure that originates at $y = 80$ and which crosses the horizontal axis at $x = 320$. Under Plan B, Luis budget line, BL_B, is $0.1x+y = 40$, or $y = 40-0.1x$, as illustrated in figure 2.11 by the dashed line that originates at $y = 40$ and crosses the horizontal axis at $x = 400$. These two budget lines intersect each other at $0.25x+(40-0.1x) = 80$, i.e., $x = 266.67$. Hence,

$$y = 40 - 0.1x = 40 - (0.1 \times 266.67) = 13.33$$

Therefore, BL_A and BL_B intersect at bundle (266.67, 13.33).

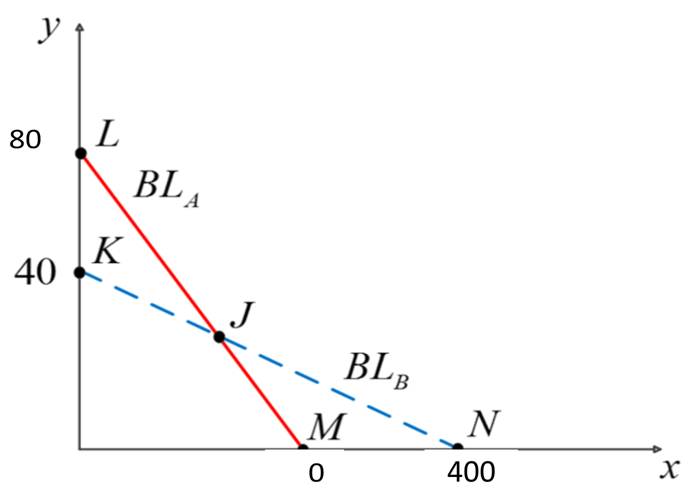

Figure 2.11. Checking WARP

3. b. If Luis mentions that Plan A is better for him, what is the set of baskets he may purchase if his behavior is consistent with the WARP?

   - According to WARP, if the consumption bundle under new prices and wealth was affordable under the original prices and wealth, $p \cdot x(p', w') \leq w$, then the bundle selected under the old prices and wealth cannot be affordable under the new prices and wealth, i.e., $p' \cdot x(p, w) \leq w'$.
   - In this context, where the consumer moves from facing budget line BL_B to BL_A, WARP states that, if the consumption bundle under BL_B, $x(p', w')$, is affordable under BL_A, it must lie on segment KJ in figure 2.11, i.e., this is equivalent to the premise of WARP, $p \cdot x(p', w') \leq w$. Hence, the bundle selected when facing budget line BL_A, $x(p, w)$, must be unaffordable under BL_B; that is, $x(p, w)$ must lie on segment LJ of budget line BL_A. Notice that bundles in segment JM are instead affordable under BL_B, thus violating WARP.

5. **Question 5 [25 Points].**Consider an industry with 2 firms competing a la Cournot. They face a constant elasticity of substitution (CES) inverse demand function $p(q) = 8q^{-\frac{1}{2}}$, and are symmetric in their constant marginal cost $c > 0$, and no fixed costs.

   (a) Find every firm i's best response function.

   - Let q_1 be the output of firm 1, and q_2 be the aggregate output of firm 2, then every firm i solves the following profit maximization problem.

$$\begin{aligned}\max_{q_i \geq 0} & \, p(q)\, q_i - cq_i \\ &= \left(8\left(q_1 + q_2\right)^{-\frac{1}{2}} - c\right) q_1\end{aligned}$$

   Taking the first order condition with respect to q_i,

$$8\left(q_1 + q_2\right)^{-\frac{1}{2}} - c - 4\left(q_1 + q_2\right)^{-\frac{3}{2}} q_i \leq 0 \quad \text{with equality if} \quad q_1 > 0$$

   Multiplying both sides by $(q_1 + q_2)^{\frac{3}{2}}$, and assuming interior solutions, we have an implicit expression for firm i's best response function.

$$4q_1 + 8q_2 = c\left(q_1 + q_2\right)^{\frac{3}{2}}$$

   (b) Find equilibrium output, prices, and profits.

   - *Equilibrium output.* In a symmetric equilibrium, output satisfies $q_1 = q_2$. Therefore, we can express $q_1 = q_i^*$ so that the equilibrium output, q_i^*, solves

$$4q_i + 8q_i = c\left(2q_i\right)^{\frac{3}{2}}$$

   which simplifies to

$$12q_i = c2^{\frac{3}{2}} q_i^{\frac{3}{2}}$$

   and after rearranging, we have

$$q_i^* = 2^{-3}\left[\frac{12}{c}\right]^2$$

   - *Equilibrium price.* Substituting the equilibrium output, q_i^*, into the inverse demand function, market price becomes

$$\begin{aligned}p\left(q^*\right) &= 8\left(2q_i^*\right)^{-\frac{1}{2}} \\ &= 8\left(\frac{12}{2c}\right)^{-1} \\ &= \frac{4c}{3}\end{aligned}$$

   - *Equilibrium profits.* Lastly, the equilibrium profit earned by every firm i becomes

$$\begin{aligned}\pi_i\left(q_i^*\right) &= \left(p\left(q^*\right) - c\right) q_i^* \\ &= \frac{2^{-3}c}{3}\left[\frac{12}{c}\right]^2 \\ &= \frac{48}{2^3 c} = \frac{6}{c}\end{aligned}$$

   (c) Consider now a cartel agreement between these two firms. Which is the cartel output that each firm should produce to maximize joint (cartel) profits?

- The cartel would choose the aggregate output level that solves the following profit maximization problem:

$$\max_{q \geq 0} p(q)\, q - cq$$
$$= \left(8q^{-\frac{1}{2}} - c\right) q$$

Taking the first order condition with respect to q,

$$8q^{-\frac{1}{2}} - c - 4q^{-\frac{1}{2}} \leq 0 \quad \text{with equality if} \quad q > 0$$

Assuming interior solutions, the cartel output becomes

$$q^c = \frac{16}{c^2}$$

Therefore, every firm i would produce $\frac{1}{2}$ share of this aggregate output, which yields an individual output of $q_i^c = \frac{8}{c^2}$.

GOOD LUCK!