Midterm #1 EconS 501

Friday, October 5th, 2018

NAME: ____________________

Instructions. *Show all your work clearly and make sure you justify all your answers.*

1. **Question 1 [15 Points].** Let us consider the case of two goods, $L = 2$. Then, an individual prefers a bundle $x = (x_1, x_2)$ to another bundle $y = (y_1, y_2)$ if and only if it contains more units of both goods than bundle y, i.e. $x_1 \geq y_1$ and $x_2 \geq y_2$. Informally, *he prefers more of everything.* Check if this preference relation satisfies: (1) completeness, (2) transitivity and (3) strong monotonicity. Assume a bundle equal to $(4, 2)$ and use a graph to support your answer.

Solution

Completeness: First, the upper contour set (UCS) describes the region of bundles that the consumer prefers to bundle $(4, 2)$. The UCS of bundle $(4, 2)$ thus contains those bundles (x_1, x_2) with weakly more than two units of good 1 and weakly more than one unit of good 1, that is

$$UCS(4, 2) = \{(x_1, x_2) \succsim (4, 2) \Leftrightarrow x_1 \geq 4 \text{ and } x_2 \geq 2\}$$

as depicted in the shaded region of figure 1 labeled UCS, which lies to the northeast $(4, 2)$.

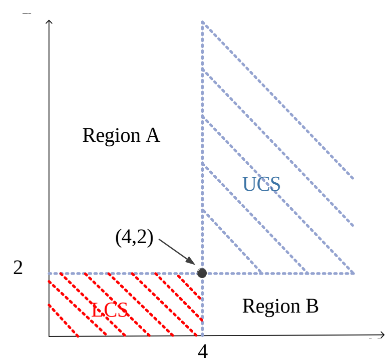

Figure 1

In contrast, the bundles in the lower contour set of bundle $(4, 2)$, whereby the consumer is better off with bundle $(4, 2)$, contain fewer units of both goods, that is,

$$LCS(4, 2) = \{(4, 2) \succsim (x_1, x_2) \Leftrightarrow x_1 \leq 4 \text{ and } x_2 \leq 2$$

The lower contour set (LCS) is therefore illustrated by those points to the southwest of $(4, 2)$; as depicted in figure 1.

Finally, the indifference set, IND $(4, 2)$, which comprises those bundles (x_1, x_2) for which the consumer is indifferent between $(4, 2)$ and (x_1, x_2), is empty, since there are no regions for which the upper contour set and the lower contour set overlap.

Hence, it is not complete, since completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both).

Transitivity: It is transitive, since transitivity requires that, for any three bundles x, y and z, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$. Now $x \succsim$y and $y \succsim z$ means that $x \geq y$ and $y \geq z$.

In vector notation, this means that bundle x is weakly larger than y in every component. Similarly, bundle y is weakly larger than bundle z in every component. Hence,

$x \geq y$ and $y \geq z$, which implies that $x_1 \geq y_1$ and $y_1 \geq z_1$ for every good l.

Strong monotonicity: It is strongly monotone. The property of strong monotonicity requires that if we increase one of the goods in a given bundle, then the newly created bundle must be strictly preferred to the original bundle. More compactly, for a given bundle y, if bundle x satisfies

$$x \geq y \text{ and } x \neq y \text{ then } x \succ y$$

In words, bundle x being weakly larger than y in all components (but being strictly larger in at least one component, since they cannot completely coincide) implies that bundle x is weakly preferred to y, and it can never be that bundle y is weakly preferred to bundle x. We can therefore conclude that x is strictly preferred to y, $x \succ y$, as required.

2. **Question 2 [20 Points].** Let $(\mathcal{B}, C(\cdot))$ be a choice structure where $\mathcal{B}$ includes all non-empty subsets of consumption bundles X, i.e., $C(B) \neq \varnothing$ for all sets $B \in \mathcal{B}$. We define the choice rule $C(\cdot)$ to be *distributive* if, for any two sets B, B' and B'' in $\mathcal{B}$,

$$C(B) \cup [C(B') \cap C(B'')] \neq \varnothing \text{ implies that } C(B) \cup [C(B') \cap C(B'')] = C(B \cup (B' \cap B''))$$

Show that, if choice rule $C(\cdot)$ is *distributive*, then choice structure $(\mathcal{B}, C(\cdot))$ does not necessarily satisfy the weak axiom of revealed preference. Consider the consumption set $X = \{x, y, z\}$ (A counterexample suffices.)

One possible counterexample is with the consumption set $X = \{x, y, z\}$ and family of budget sets

$$\mathcal{B} = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$$

Let the choice rule $C(\cdot)$ be given by

$$C\{x\} = \{x\},\ C\{y\} = \{y\} \ \text{ and } C\{z\} = \{z\},$$

when facing a single available element,

$$C\{x, y\} = \{x, y\},\ C\{x, z\} = \{x, z\},\ C\{y, z\} = \{y, z\}$$

when facing two available elements, and

$$C\{x, y, z\} = \{x\}$$

when facing all three elements. First, note that this choice rule is distributive. In particular, the next list considers all possible pairs of budget sets.

$$\begin{aligned}
C(x) \cup [C(\{y\}) \cap C(\{z\})] &= \{x\} \cup \{\varnothing\} = \{x\} = C(\{x\}) \\
C(x) \cup [C(\{y\}) \cap C(\{x, z\})] &= \{x\} \cup \{\varnothing\} = \{x\} = C(\{x\}) \\
C(x) \cup [C(\{y\}) \cap C(\{x, y, z\})] &= \{x\} \cup \{\varnothing\} = \{x\} = C(\{x\}) \\
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C(z) \cup [C(\{x\}) \cap C(\{y\})] &= \{z\} \cup \{\varnothing\} = \{z\} = C(\{z\}) \\
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C(z) \cup [C(\{x,z\}) \cap C(\{y,z\})] &= \{z\} \cup \{z\} = \{z\} = C(\{z\}).
\end{aligned}
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However, note that the weak axiom is not satisfied. In particular, while x and y both belong to $\{x,y\}$ and to $\{x,y,z\}$, this individual selects $C\{x,y\} = \{x,y\}$ (and does not select x) but changes his choice to x (and not y) when his set of available options expands to include z, i.e., $C\{x,y,z\} = \{x\}$. Thus, the weak axiom fails.

3. **Question 3 [25 Points].** An individual consumes only good 1 and 2, and his preferences over these two goods can be represented by Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta} \quad \text{where } \alpha, \beta > 0.$$

Assume $\alpha + \beta = 1$. This individual currently works for a firm in a city where initial prices are $p^0 = (p_1, p_2)$, and his wealth is w.

(a) Find the Walrasian demand for goods 1 and 2 of this individual, $x_1(p, w)$ and $x_2(p, w)$. Show that it satisfies homogeneity of degree zero.

- We know that the Walrasian demand in the Cobb-Douglas case are $x_1(p, w) = \frac{\alpha w}{p_1}$ and $x_2(p, w) = \frac{\beta w}{p_2}$; and it satisfies homogeneity of degree zero since $x_1(\lambda p, \lambda w) = \frac{\alpha \lambda w}{\lambda p_1} = \frac{\alpha w}{p_1} = x_1(p, w)$.

(b) Find his indirect utility function at price vector p, and denote it as $v(p, w)$.

- Plugging the above Walrasian demand functions in the consumer's utility function, we obtain

$$
\begin{aligned}
v(p, w) &= \left[\frac{\alpha w}{p_1}\right]^{\alpha} \left[\frac{\beta w}{p_2}\right]^{\beta} \\
&= w \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{\beta}{p_2}\right)^{\beta}
\end{aligned}
$$

(c) The firm that this individual works for is considering moving its office to a different city, where good 1 has the same price, but good 2 (e.g., housing) is twice as expensive, i.e., the new price vector is $p' = (p_1, 2p_2)$. Find the value of the indirect utility function in the new location. Let us denote this indirect utility function $v(p', w)$.

- The indirect utility function $v(p', w)$ is

$$v(p', w) = w \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{\beta}{2p_2}\right)^{\beta}$$

where, relative to $v(p, w)$, only the price of good 2 has changed (namely, it has doubled), while all other elements remain unaffected.

(d) This individual's expenditure function is[1]

$$e(p,u) = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

Evaluate this expenditure function in the following cases:

1. Under initial prices, p, and maximal utility level $u \equiv v(p,w)$, and denote it by $e(p,u)$.

$$e(p,u) = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\underbrace{\left[w\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\right]}_{u} = w$$

2. Under initial prices, p, and maximal utility level $u' \equiv v(p',w)$, and denote it by $e(p,u')$.

$$e(p,u') = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left[w\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{2p_2}\right)^{\beta}\right] = \frac{1}{2^{\beta}}w$$

3. Under new prices, p', and maximal utility level $u \equiv v(p,w)$, and denote it by $e(p',u)$.

$$e(p',u) = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{2p_2}{\beta}\right)^{\beta}\left[w\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\right] = 2^{\beta}w$$

4. Under new prices, p', and maximal utility level $u' \equiv v(p',w)$, and denote it by $e(p',u')$.

$$e(p',u') = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{2p_2}{\beta}\right)^{\beta}\left[w\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{2p_2}\right)^{\beta}\right] = w$$

(e) Find this individual's equivalent variation due to the price change. Explain how your result can be related with this proposal of the worker to his boss: "I would really prefer to stay in this city. In fact, I would accept a salary reduction if I could keep working for the firm in this city."

- The equivalent variation of a price change is given by

$$EV = e(p',u') - e(p,u')$$

using the results from the previous part, we have that $e(p',u') = w$, while $e(p,u') = \frac{1}{2^{\beta}}w$, thus implying that the equivalent variation is

$$EV = w - \frac{1}{2^{\beta}}w$$

That is, this individual would be willing to accept a reduction in his wealth of $w - \frac{1}{2^{\beta}}w$ in order to avoid moving to a different city. [Alternatively, the individual is willing to accept a reduction of $\left(1-\frac{1}{2^{\beta}}\right)\%$ of his wealth.] Figure 1 depicts the equivalent variation for the case in which $\alpha = \beta = \frac{1}{2}$, i.e., $EV = w\left(1 - \frac{1}{\sqrt{2}}\right)$. In particular, the figure depicts the wealth level of this individual, w, in the 45^0-line; and the equivalent variation (in the shaded area). Hence, the unshaded region below the 45^0-line represents the remaining income that this

[1] As a practice, you can set up the consumer's expenditure minimization problem (EMP), find the Hicksian demands that emerge from solving this EMP, $h_1(p,u)$ and $h_2(p,u)$, and afterwards plug them into $p_1x_1 + p_2x_2$ to obtain the expenditure function $e(p,u) \equiv p_1h_1(p,u) + p_2h_2(p,u)$. After some algebra, you should find an expression of $e(p,u)$ that coincides with that provided in the exercise.

individual would retain after giving up the amount found in the equivalent variation.

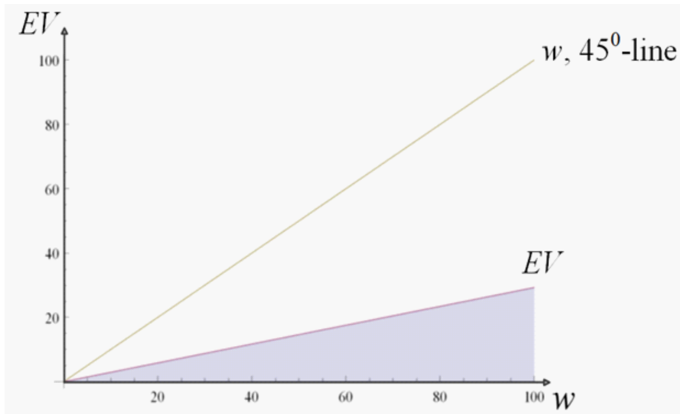

Figure 1. Equivalent variation (shaded area)..

(f) How is this individual's consumer surplus affected by the price change? (The change in consumer surplus is often referred to as the "area variation (AV)"

- The area variation is given by the area below the Walrasian demand of good 2 (since only the price of this good changes), between the initial and final price level. That is,

$$AV = \int_{p_2}^{2p_2} x_2(p, w)dp = \int_{p_2}^{2p_2} \frac{\beta w}{p} dp$$

and rearranging

$$= \beta w \int_{p_2}^{2p_2} \frac{1}{p} dp = \beta w \ln 2$$

Hence, moving to the new city would imply a reduction in this individual's welfare of $\beta w \ln 2$, or $(\beta \ln 2)\,\%$ of his wealth. Figure 2 depicts the AV for the case in which $\alpha = \beta = \frac{1}{2}$, i.e., $AV = \frac{\ln 2}{2} w$, and compares it with the EV found in part (e) of the exercise.

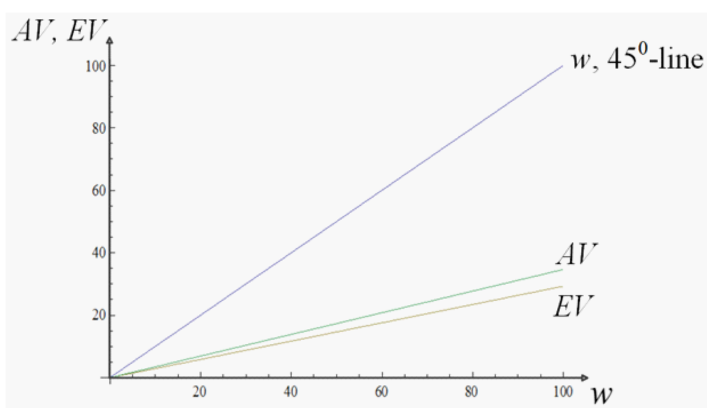

Figure 2. Area variation and equivalent variation.

(g) Which of the previous welfare measures in questions (e) and (f) coincide? Which of them do *not* coincide? Explain.

- None of them coincide, since this individual's preferences produces a positive income effect.

(h) Consider how the welfare measures from questions (e) and (f) would be modified if this individual's preferences were represented, instead, by the utility function $v(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$. You do not need to do any calculations to answer this question. A verbal explanation suffices.

- Since we have just applied a monotonic transformation to the initial utility function, $u(x_1, x_2)$, the new utility function $v(x_1, x_2)$ represents the same preference relation as utility function $v(x_1, x_2)$. Hence, the welfare results that we would obtain from function $v(x_1, x_2)$ would be the same as those with utility function $u(x_1, x_2)$. This is, in fact, one of the advantages of using monetary measures of welfare change (such as the equivalent, compensating, or area variation) rather than the simple difference in utility levels before and after the price change, i.e., $u' - u$. In particular, while the monetary measures are insensitive to monotonic transformations of the utility function, the utility difference when the consumer has utility function $u(x)$, i.e., $u' - u$, may differ from that when his utility experiences a monotonic transformation, $v' - v$.

4. **Question 4 [25 Points].** Consider that your closest friend does not work and he currently receives help from his family, i.e., a fixed salary $w > 0$ which does not adjust to inflation. His expenditure function is

$$e(p_x, p_y, u) = \sqrt{(p_x + p_y)u},$$

where $p_x, p_y > 0$ denote initial prices. Suppose that prices of goods x and y increase to p'_x and p'_y, respectively.

(a) You want to give him a monetary gift so that he will not be affected by the above price increase. How much money should you give him? That is, find his compensating variation (CV).

- From duality, we know that $e(p_x, p_y, u) = w$, which helps us rewrite the above equation as $w = \sqrt{(p_x + p_y)}u$. Therefore, at the initial price vector, the maximum utility the retiree can obtain is $u = \frac{w}{\sqrt{p_x+p_y}}$. In order to ensure that he is not worse off after the price increase, we need to give him an amount of money that covers the difference between the new expense and the old expense, that is,

$$\begin{aligned} CV &= w' - w \\ &= e(p'_x, p'_y, u) - w \\ &= \underbrace{\sqrt{(p'_x + p'_y)}u}_{e(p'_x, p'_y, u)} - w \\ &= \sqrt{(p'_x + p'_y)}\underbrace{\frac{w}{\sqrt{p_x + p_y}}}_{u} - w \\ &= w\left(\frac{\sqrt{(p'_x + p'_y)}}{\sqrt{p_x + p_y}} - 1\right) \end{aligned}$$

(b) Now find his equivalent variation (EV) from the price change, i.e., the change in income needed at initial prices p_x and p_y that would have the same effect on utility as would the change in prices, p'_x and p'_y.

- The equivalent variation is

$$\begin{aligned} EV &= e(p_x, p_y, u') - e(p'_x, p'_y, u') \\ &= (p_1 + p_2)u' - (p'_1 + p'_2)u' \\ &= \sqrt{p_x + p_y}\underbrace{\frac{w}{\sqrt{(p'_x + p'_y)}}}_{u'} - \sqrt{(p'_x + p'_y)}\underbrace{\frac{w}{\sqrt{(p'_x + p'_y)}}}_{u'} \\ &= w\left(\frac{\sqrt{p_x + p_y}}{\sqrt{(p'_x + p'_y)}} - 1\right) \end{aligned}$$

(c) Which is larger in this case, CV or EV?

- In this case, since $p'_x + p'_y > p_x + p_y$, we obtain that the compensating variation is positive $CV > 0$, but the equivalent variation is negative $EV < 0$, entailing that $CV > 0 > EV$.

(d) Find his Walrasian demand for each good.

- *Good x.* From duality, we know that $e(p_x, p_y, u) = w$, which yields $w = \sqrt{p_x + p_y}u$. Solving for u, we find the indirect utility function

$$v(p_1, p_2) = \frac{w}{\sqrt{p_x + p_y}}$$

Then, we can insert this indirect utility function into Roy's identity, as follows, which yields the Walrasian demand for good x:

$$x(p_x, p_y, w) = -\frac{\frac{\partial v(p_x,p_y)}{\partial p_x}}{\frac{\partial v(p_x,p_y)}{\partial w}} = -\frac{-\frac{1}{2}\frac{w}{\sqrt{p_x+p_y}(p_x+p_y)}}{\frac{1}{\sqrt{p_x+p_y}}} = \frac{1}{2}w(p_x + p_y)^{-1}.$$

- *Good y.* Following a similar approach, we can find the Walrasian demand for good y:

$$y(p_x, p_y, w) = -\frac{\frac{\partial v(p_1,p_2)}{\partial p_2}}{\frac{\partial v(p_1,p_2)}{\partial w}} = -\frac{-\frac{1}{2}\frac{w}{\sqrt{p_x+p_y}(p_x+p_y)}}{\frac{1}{\sqrt{p_x+p_y}}} = \frac{1}{2}w(p_x + p_y)^{-1}$$

implying that the consumer purchases the same amount of both goods, $x(p_x, p_y, w) = y(p_x, p_y, w)$.

(e) Find his utility function. What is this type of utility function called?

- In part (d), we found that the consumer in this exercise, purchases the same amount of both goods, $x(p_x, p_y, w) = y(p_x, p_y, w)$, regardless of the price vector and income he faces. This only occurs when the consumer regards both goods as complements, exhibiting a Leontief utility function $u(x, y) = A\min\{x, y\}$, where $A > 0$. As a remark, note that we do not say that his utility function is the general expression of the Leontief utility function $u(x, y) = A\min\{ax, by\}$, where $A, a, b > 0$ since in such a setting the consumer could purchase different amounts of each good; as long as he keeps the proportion of goods he consumes constant.

5. **Question 5 [15 Points].** Verbally and graphically discuss why the Giffen paradox from consumer theory cannot arise in production theory.

Check lecture notes.

GOOD LUCK!