Homework #4 - EconS 527 (Due on 10/08

1. onsider the following simultaneous-move game between player 1 (in rows) and player 2 (in columns).

		Player 2		
		L	M	R
Player 1	U	$4,2$	$2,0$	$0,3$
	C	$5,1$	$3,2$	$\delta,4$
	D	$5,2$	$6,2$	α,β

We are informed that player 1 finds that strategy C weakly dominates U. In addition, we are told that strategy profile (D,R) is a Nash equilibrium but (D,M) is not. Using this information, answer the following questions.

Before we start, let us define $u_i(s_i,s_j)$ to be the utility function of player i when player i deploys strategy $s_i \in S_i$ and his counterpart, player j, deploys strategy $s_j \in S_j$, where $i,j=\{1,2\}$. In this context, $S_1=\{U,C,D\}$ and $S_2=\{L,M,R\}$.

(a) Does player 2 have a strictly dominated strategy?

- Yes, strategy M is strictly dominated by R for player 2. From the fact that strategy profile (D,R) is a Nash equilibrium but (D,M) is not, we can infer that $u_2(D,R) > u_2(D,M)$, which implies that $\beta > 2$. Note that if $\beta = 2$, then

$$\begin{aligned} 6 &= u_1(D,M) > u_1(C,M) = 3, \text{ and} \\ 6 &= u_1(D,M) > u_1(U,M) = 2. \end{aligned}$$

or more compactly,

$$u_1(D,M) > \max\{u_1(C,M), u_1(U,M)\}$$

so that (D,M) would also be a Nash equilibrium, that is contradicting our premise. Therefore,

$$u_2(s_1,R) > u_2(s_1,M)$$

for every strategy of player 1 $s_1 \in S_1 = \{U,C,D\}$.

(b) Is strategy profile (D,R) the unique Nash equilibrium of this game?

- After strategy M is removed, the normal form representation of the game becomes

		Player 2	
		L	R
Player 1	U	$4, 2$	$0, 3$
	C	$5, 1$	$\delta, 4$
	D	$5, 2$	α, β

No, strategy profile (D, R) may not be the unique Nash equilibrium of this game. From (D, R) being a Nash equilibrium, $\alpha \geq \delta$. This entails that other Nash equilibria can be supported. Let us now find all Nash equilibria of this game by identifying each player's best response function (BRF).

- The next payoff matrix underlines best response payoffs in the case that $\alpha = \delta$ (left panel) and in the case that $\alpha > \delta$ (right panel).

If $\alpha = \delta$:

		Player 2	
		L	R
Player 1	U	$4, 2$	$0,\underline{3}$
	C	$\underline{5}, 1$	$\underline{\delta},\underline{4}$
	D	$\underline{5}, 2$	$\underline{\alpha},\underline{\beta}$

If $\alpha > \delta$:

		Player 2	
		L	R
Player 1	U	$4, 2$	$0,\underline{3}$
	C	$\underline{5}, 1$	$\delta,\underline{4}$
	D	$\underline{5}, 2$	$\underline{\alpha},\underline{\beta}$

For player 1, $BR_1(L) = \{C, D\}$ and $BR_1(R) = \{C, D\}$ if $\alpha = \delta$ but $BR_1(R) = \{D\}$ if $\alpha > \delta$. Whereas, for player 2, $BR_2(U) = \{R\}$, $BR_2(C) = \{R\}$, and $BR_2(D) = \{R\}$. Therefore, if $\alpha = \delta$, strategy profiles (C, R) and (D, R) both support a Nash equilibrium. However, when $\alpha > \delta$, strategy profile (D, R) is the unique Nash equilibrium.

(c) Does player 1 have a strictly dominant strategy?

- No, player 1 does not have a strictly dominant strategy. First, we are informed that player 1 finds that strategy C weakly dominates U, that is, $\delta \geq 0$. In addition, $u_1(D, L) = u_1(C, L) = 5$ implies that strategy D does not strictly dominate strategy C for player 1, which holds for any parameter values of α and δ. Finally, strategy D does not strictly dominate strategy U when $\alpha = 0$.

2. Consider a sequential-move bargaining game between Player 1 (proposer) and Player 2 (responder). Player 1 makes a take-it-or-leave-it offer to Player 2, specifying an amount $s = \{0, \frac{1}{2}v, v\}$ out of an initial surplus v, i.e., no share of the pie, half of the pie, or all of the pie. If Player 2 accepts such a distribution Player 2 receives the offer s, while Player 1 keeps the remaining surplus $v - s$. If Player 2 rejects, both players get a zero payoff.

(a) Describe the strategy space for every player.

- Strategy set for player 1 is

$$S_1 = \{0, \frac{1}{2}v, v\}$$

while the strategy set for player 2 is

$$S_2 = \{AAA, AAR, ARR, RRR, RRA, RAA, ARA, RAR\}$$

For every triplet, the first component specifies player 2's response upon observing that Player 1 makes an offer $s = v$; the second component is his response to an offer $s = \frac{1}{2}v$; and the third component describes player 2's response to an offer $s = 0$.

(b) Provide the normal-form representation of this bargaining game.

- Using the three strategies for Player 1 and the eight available strategies for Player 2, the 3×8 matrix below represents the normal-form representation of this game

		Player 2							
		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
	$s = v$	$0, v$	$0, v$	$0, v$	$0, v$	$0, 0$	$0, 0$	$0, 0$	$0, 0$
Player 1	$s = \frac{1}{2}v$	$\frac{1}{2}v, \frac{1}{2}v$	$\frac{1}{2}v, \frac{1}{2}v$	$0, 0$	$0, 0$	$\frac{1}{2}v, \frac{1}{2}v$	$\frac{1}{2}v, \frac{1}{2}v$	$0, 0$	$0, 0$
	$s = 0$	$v, 0$	$0, 0$	$v, 0$	$0, 0$	$v, 0$	$0, 0$	$v, 0$	$0, 0$

(c) Does any player have strictly dominated pure strategies?

- No player has any strictly dominated pure strategy:
- *Player 1.* For player 1, we find that $s = \frac{1}{2}v$ yields a weakly (not strictly) higher payoff than $s = v$, that is $u_1(s = \frac{1}{2}v, s_2) \geq u_1(s = v, s_2)$ for all strategies of player 2, $s_2 \in S_2$ (i.e., some columns in the above matrix), which is satisfied with strict equality for some strategies of player 2, such as ARR, RRR or RRA.
- Similarly, $s = 0$ yields a weakly (but not strictly) higher payoff than $s = v$. That is, $u_1(s = 0, s_2) \geq u_1(s = v, s_2)$ for all $s_2 \in S_2$, with strict equality for some $s_2 \in S_2$, such as ARR and RRR.
- Finally, $s = \frac{1}{2}v$ yields a higher payoff than $s = 0$ against some strategies of player 2, such as AAR, but a lower payoff against other strategies, such as AAA and RAR. Hence, there is no weakly dominated strategy for Player 1.

- *Player 2.* Similarly, for Player 2, $u_2(s_2, s_1) \geq u_2(s_2', s_1)$ for any two strategies of Player 2 $s_2 \neq s_2'$ and for all $s_1 \in S_1$ with strict equality for some $s_1 \in S_1$.

(d) Does any player have strictly dominated mixed strategies?

- Once we have shown that there is no strictly dominated pure strategy, we focus on the existence of strictly dominated *mixed* strategies. We know that player 1 is never going to mix assigning a strictly positive probability to his pure strategy $s = v$ (i.e., offering the whole pie to Player 2) given that it will reduce for sure his expected payoff, for any strategy with which player 2 responds. Indeed, since such strategy yields a strictly lower (or equal) payoff than other of his available strategies, such as $s = 0$ or $s = v/2$.
- If Player 1 mixes between $s = 0$ and $s = v/2$, we can see that he is going to obtain a mixed strategy σ_1 that yields an expected utility, $u_1(\sigma_1, s_1)$, which exceeds his utility from selecting the pure strategy $s = v$. That is,

$$u_1(\sigma_1, s_1) \geq u_1(s = v, s_2) \text{ for all } s_2 \in S_2$$

with strict equality for $s_2 = ARR$ and $s_2 = RRR$, but strict inequality (yielding a strictly higher expected payoff) for all other strategies of player 2. We can visually check this result in the above normal-form representation by noticing that $s = v$, in the top row, yields a zero payoff for any strategy of player 2. However, a linear combination of strategies $s = v/2$ and $s = 0$, in the middle and bottom rows, yields a positive expected payoff for columns AAA, AAR, RRA, RAA, ARA and RAR; since all of them contain at least one positive payoff for player 1 in the middle or bottom row. However, in the remaining columns (ARR and RRR), player 1's payoff is zero both in the middle and bottom row; thus implying that his expected payoff, zero, coincides with his payoff from playing the pure strategy $s = v$ in the top row. A similar argument applies to Player 2. Therefore, there is no strictly dominated mixed strategy.

3. Consider a Cournot duopoly with linear inverse demand curve $p(q) = a - q$, where q denotes aggregate output. Both firms have a common constant marginal cost $c > 0$, and where $a > c$. Assume that firms do an equity swap of γ, i.e., each firm i receives a share $0 < \gamma \leq \frac{1}{2}$ in firm j's profits, where $j \neq i$.

   (a) Find the Cournot equilbrium output, (q_1^C, q_2^C).

- Firm i's profit-maximization problem (PMP) is given by

$$\max_{q_i} \underbrace{(1-\gamma)(a-q_i-q_j-c)q_i}_{\text{Firm } i\text{'s profit}} + \underbrace{\gamma(a-q_i-q_j-c)q_j}_{\text{Firm } j\text{'s profit}}$$

  Taking first-order conditions with respect to q_i yields

$$(1-\gamma)(a-q_i-q_j-c)-(1-\gamma)q_i-\gamma q_j=0$$

  In a symmetric equilibrium $q_i^C = q_j^C = q^C$, which lets us simplify the above expression as follows

$$(1-\gamma)(a-2q^C-c)-(1-\gamma)q^C-\gamma q^C=0$$

  which, solving for q^C yields

$$q^C = \frac{(1-\gamma)(a-c)}{3-2\gamma}$$

(b) Evaluate equilibrium output q_i^C at $\gamma = 0$ and $\gamma = \frac{1}{2}$. Interpret.

- When firms do not benefit from each other's profits, $\gamma = 0$, equilibrium output q^C becomes $\frac{a-c}{3}$, thus coinciding with that under the standard Cournot model with linear inverse demand curve $p(q) = a - q$. In contrast, when firms fully share their profits, $\gamma = 1/2$, equilibrium output q^C becomes $\frac{a-c}{4}$, thus coinciding with half of monopoly output (or cartel output).

(c) Determine if q_i^C increases or decreases in γ.

- Differentiating q^C with respect to share γ yields

$$\frac{\partial q^C}{\partial \gamma} = -\frac{a-c}{3-2\gamma} + \frac{2(1-\gamma)(a-c)}{(3-2\gamma)^2}$$

  which simplifies to

$$-\frac{a-c}{(3-2\gamma)^2}$$

  which is clearly negative since $a > c$ by definition. Intuitively, as firms share more of each other's profits, their individual PMP resembles the joint PMP in a cartel, leading each of them to reduce its production.

- For illustrative purpose, figure 1 depicts q^C as a function of the profit share γ.

For simplicity, we consider $a = 1$ and $c = 0$ which yields a output $q^C = \frac{1-\gamma}{3-2\gamma}$.

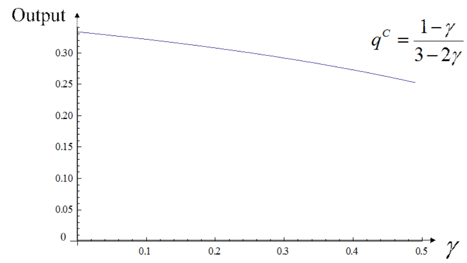

Figure 1. Output q^C as a function of γ

(d) Find equilibrium profits, π^C, and determine whether they increase or decrease in γ.

- Equilibrium profits are $\pi^C = \frac{(1-\gamma)(a-c)^2}{(3-2\gamma)^2}$ which increase in γ since

$$\frac{\partial \pi^C}{\partial \gamma} = \frac{(1-2\gamma)(a-c)^2}{(3-2\gamma)^3}$$

  is positive given that $\gamma \leq \frac{1}{2}$ by definition. Intuitively, as firms take more into account each other's profits, their individual profits approach those they would obtain under a cartel agreement, which are larger than under a standard Cournot model.

- Figure 2 plots π^C where, similarly as for the equilibrium output, we consider parameter values $a = 1$ and $c = 0$.

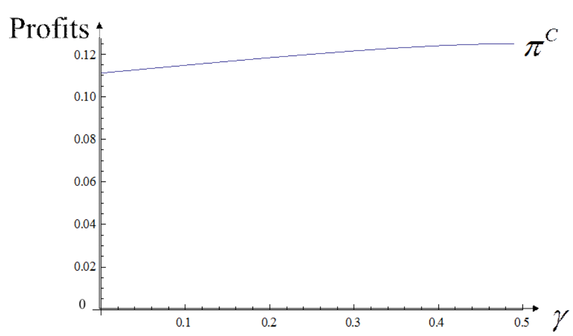

Figure 2. Profits π^C as a function of γ.

4. Consider a Cournot duopoly where firms sell their production in a competitive market (e.g., an international market where the duopolists' sales represent a small share of total sales) at prices $p_1 = \$2$ and $p_2 = \$3$. Both firms face a concave supply function,

$q_1 = 13x_1 - 0.2x_1^2$ for firm 1 and $q_2 = 12x_2 - 0.1x_2^2$ for firm 2, where x_1 denote the amount of input (e.g., labor) that firm 1 hires; and similarly for x_2, which indicates the amount of input that firm 2 hires. Assume that the cost of hiring x_i units of input for firm i when its rival hires x_j units of that input is $C(x_i, x_j) = [2 + 0.1(x_i + \beta x_j)]\, x_i$ where β represents the cost externality that firm i suffers from every unit of input firm j hires. Specifically, when $\beta = 0$, the above cost function collapses to $C(x_i, x_j) = [2 + 0.1x_i]\, x_i$, thus being independent on firm j's hiring decisions. In contrast, when $\beta > 0$, firm j's hiring decisions increase firm i's cost. For instance, high skill workers may become more scarce, and thus firm i needs to offer them a higher salary to attract them to work for firm i.

(a) Write down firm 1's profit-maximization problem. Find this firm's best response function. Evaluate it at $\beta = 0$ and at $\beta > 0$. Interpret. Repeat your analysis for firm 2.

- *Firm 1.* Firm 1 chooses its input level x_1 to solve

$$\max_{x_1 \geq 0} \pi_1 = 2q_1 - C(x_1, x_2) = 2\underbrace{(13x_1 - 0.2x_1^2)}_{q_1} - [2 + 0.1(x_1 + \beta x_2)]x_1$$

  Differentiating with respect to input level x_1, yields

$$\frac{\partial \pi_1}{\partial x_1} = -2 + 2(13 - 0.4x_1) - 0.1x_1 - 0.1(x_1 + \beta x_2)$$

  Rearranging, we find firm 1's best response function

$$x_1(x_2) = 24 - 0.1\beta x_2 \qquad (BRF_1)$$

  This best response function decreases in firm 2's output, x_2, and in the cost externality that firm 1 suffers from firm 2, β. Graphically, an increase in the cost externality (larger β) pivots firm 1's best response function BRF_1 inwards, leaving its vertical unaffected at 24. Intuitively, firms' output increase their strategic substitutability (as captured by the negative slope of the best response function) when the cost externality becomes more severe.

- *Firm 2.* Firm 2 chooses its input level x_2 to solve

$$\max_{x_2 \geq 0} \pi_2 = 3q_2 - C(x_1, x_2) = 3\underbrace{(12x_2 - 0.1x_2^2)}_{q_2} - [2 + 0.1(\beta x_1 + x_2)]x_2$$

Differentiating with respect to input level x_2, yields

$$\frac{\partial \pi_2}{\partial x_2} = -2 + 3(12 - 0.2x_2) - 0.1x_2 - 0.1(\beta x_1 + x_2) = 0$$

Rearranging, we find firm 2's best response function

$$x_2(x_1) = 42.5 - 0.125\beta x_1 \qquad (BRF_2)$$

Firm 2's output is also decreasing in the cost externality that firm 1 produces, as captured by parameter β. A similar interpretation as for firm 1's best response function applies in this case.

(b) Determine the equilibrium values of firms' hiring decisions, x_1 and x_2.

- Substituting BRF_2 into BRF_1, we obtain

$$x_1 = 24 - 0.1\beta\,[42.5 - 0.125\beta x_1]$$

Solving for x_1 yields an equilibrium input level of

$$x_1^* = \frac{1920 - 340\beta}{80 - \beta^2}$$

for firm 1. Substituting this quilibrium input level $x_1^* = \frac{1920-340\beta}{80-\beta^2}$ into BRF_2 gives

$$x_2(x_1) = 42.5 - 0.125\beta\left[\frac{1920 - 340\beta}{80 - \beta^2}\right]$$

which, solving for x_2, entails an equilibrium input level of

$$x_2^* = \frac{3400 - 240\beta}{80 - \beta^2}, \text{ for firm 2.}$$

(c) How are the equilibrium results from part (b) affected by a marginal increase in the cost externality parameter β?

$$\frac{\partial x_1^*}{\beta} = -\frac{20[1360 - \beta(192 - 17\beta)]}{(80 - \beta^2)^2} < 0$$

and,

$$\frac{\partial x_2^*}{\beta} = \frac{(6800 - 240\beta)\beta - 19200}{(80 - \beta^2)^2} < 0$$

Thus, equilibrium input levels x_1^* and x_2^* are both decreasing in the cost externality parameter β.