Homework #6 (Due on October 19th, 2018)

1. Consider an agent who exerts an effort level e, where $e \geq 0$, to generate output y that is subject to output shocks ε (e.g., weather conditions affecting the quality of harvest, machine breakdown causing product failure, etc.). Output then behaves as follows

$$y = ge + \varepsilon$$

where g denotes the agent's output efficiency, and shock ε follows a normal distribution, $N(0, \sigma^2)$, with mean of zero and variance $\sigma^2 > 0$. The agent earns a wage of $w = sy$, where $0 \leq s \leq 1$ represents his output shares (e.g., commission from sales of the products). In addition, the agent incurs a cost to exert effort e, given by

$$c(e) = \frac{1}{2}e^2$$

which is increasing and convex in his effort level e.

The agent's payoff comes from the utility of earning wage w minus his cost of effort, where

$$U = u(w) - c(e)$$

Specifically, his utility function follows the negative exponential form of

$$u(w) = 1 - \exp(-\eta w) \tag{1}$$

(a) What is the agent's Arrow-Pratt coefficient of absolute risk aversion, $r_A(w)$. How does it vary with his wage?

- The absolute risk aversion parameter is defined by $r_A(w) \equiv -\frac{u''(w)}{u'(w)}$, where

$$u'(w) \equiv \frac{\partial u(w)}{\partial w} = \eta \exp(-\eta w)$$
$$u''(w) \equiv \frac{\partial u^2(w)}{\partial w^2} = -\eta^2 \exp(-\eta w)$$

Therefore, the absolute risk aversion parameter is

$$r_A(w) \equiv -\frac{u''(w)}{u'(w)} = -\frac{-\eta^2 \exp(-\eta' w)}{\eta \exp(-\eta' w)} = \eta$$

which is independent of his wage.

(b) Find the certainty equivalent of the agent.

- The agent's expected utility from earning wage w is

$$E\left[u\left(w\right)\right] = E\left[1-\exp\left(-\eta w\right)\right] = 1-\int_{-\infty}^{+\infty}\exp\left(-\eta w\right)f\left(w\right)dw$$

Recall that the agent's wage follows a normal distribution, with density

$$f\left(w\right)=\frac{1}{\sqrt{2\pi}\sigma_w}\exp\left(-\frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2\right)$$

where μ_w denotes the expected wage, while σ_w represents its standard deviation. We can then rewrite his expected utility as

$$\begin{aligned} E\left[u\left(w\right)\right] &= 1-\int_{-\infty}^{+\infty}\exp\left(-\eta w\right)\frac{1}{\sqrt{2\pi}\sigma_w}\exp\left(-\frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2\right)dw \\ &= 1-\frac{1}{\sqrt{2\pi}\sigma_w}\int_{-\infty}^{+\infty}\exp\underbrace{\left(-\eta w-\frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2\right)}_{\text{Term } A}dw \qquad (2)\end{aligned}$$

Consider Term A inside the exponent,

$$\begin{aligned} -\eta w-\frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2 &= -\frac{w^2-2\mu_w w+2\eta\sigma_w^2 w+\mu_w^2}{2\sigma_w^2} \\ &= -\frac{w^2-2\left(\mu_w-\eta\sigma_w^2\right)w+\mu_w^2}{2\sigma_w^2} \\ &= -\frac{\left[w-\left(\mu_w-\eta\sigma_w^2\right)\right]^2+\left(2\eta\mu_w\sigma_w^2-\eta^2\sigma_w^4\right)}{2\sigma_w^2} \\ &= -\frac{\left[w-\left(\mu_w-\eta\sigma_w^2\right)\right]^2}{2\sigma_w^2}-\left(\eta\mu_w-\frac{\eta^2\sigma_w^2}{2}\right)\end{aligned}$$

Substituting the above result into expression (2), we find that the agent's expected utility from working for the firm can be expressed as

$$
\begin{aligned}
E_A\left[u\left(w\right)\right] &= 1-\frac{1}{\sqrt{2\pi}\sigma_w}\int_{-\infty}^{+\infty}\exp\left(-\frac{\left[w-\left(\mu_w-\eta\sigma_w^2\right)\right]^2}{2\sigma_w^2}-\left(\eta\mu_w-\frac{\eta^2\sigma_w^2}{2}\right)\right)dw \\
&= 1-\exp\left(-\left(\eta\mu_w-\frac{\eta^2\sigma_w^2}{2}\right)\right)\underbrace{\frac{1}{\sqrt{2\pi}\sigma_w}\int_{-\infty}^{+\infty}\exp\left(-\frac{\left[w-\left(\mu_w-\eta\sigma_w^2\right)\right]^2}{2\sigma_w^2}\right)dw}_{=1\text{ for integrating the PDF of }N(\mu_w-\eta\sigma_w^2,\sigma_w^2)} \\
&= 1-\exp\left(-\eta\left(\mu_w-\eta\frac{\sigma_w^2}{2}\right)\right) \qquad (3)
\end{aligned}
$$

Since the agent's utility function is a one-to-one mapping from wage to utility, we can define an inverse function, $u^{-1}\left(u\right)$, that maps the agent's utility to his certainty equivalent payment. A direct comparison between expressions (1) and (3) yields

$$CE\left(\mu_w,\sigma_w^2\right)=\mu_w-\eta\frac{\sigma_w^2}{2}$$

where his certainty equivalent wage, $CE\left(\mu_w,\sigma_w^2\right)$, is a function of expected wage, $\mu_w=E_A\left[w\right]$ that is decreasing in variance of output shocks σ_w^2 and risk aversion η. Specifically, his expected wage and variance are

$$
\begin{aligned}
\mu_w &= E\left[w\right]=E\left[s\underbrace{(ge+\varepsilon)}_{y}\right]=sge+s\underbrace{E\left[\varepsilon\right]}_{0}=sge \\
\sigma_w^2 &= Var\left(w\right)=Var\left(s\left(ge+\varepsilon\right)\right)=Var\left(s\varepsilon\right)=s^2\sigma^2
\end{aligned}
$$

such that his certainty equivalent wage can be equivalently expressed as

$$CE\left(e\right)=sge-\eta\frac{s^2\sigma^2}{2}$$

Intuitively, the agent would accept a lower certainty equivalent wage when output shocks, σ, becomes more volatile or when he becomes more risk averse (higher η). In contrast, he needs a higher certainty equivalent when he exerts a larger effort level e.

(c) Find the agent's optimal effort e^*. How does it vary with his output share s, output efficiency g, risk aversion η, and output shocks σ?

- The agent's expected payoff now becomes

$$\begin{aligned} E\left[U\left(e\right)\right] &= E\left[u\left(w\right)\right]-E\left[c\left(e\right)\right] \\ &= CE\left(e\right)-c\left(e\right) \\ &= sge-\eta\frac{s^{2}\sigma^{2}}{2}-\frac{1}{2}e^{2} \end{aligned}$$

Differentiating his expected payoff with respect to effort e, and assuming interior solutions, we find

$$\frac{\partial E\left[U\left(e\right)\right]}{\partial e}=sg-e=0$$

such that the optimal effort satisfies $e^{*}=sg$.

- The optimal effort e^{*} is increasing in his output share s and in output efficiency g, but is independent of risk aversion (parameter η) and the volatility of output shocks σ. Intuitively, when the agent obtains a higher share of the output, or when he becomes more efficient at producing output, he exerts more effort to generate a higher output. However, since in expectation his share of output is not affected by risk aversion or output shocks, his optimal effort is not affected by these two parameters.

2. Consider a similar setting to that in the previous exercise. However, we now analyze a worker exerting two levels of efforts, e_1 and e_2, where $0 \leq e_i \leq +\infty$ for every $i \in \{1,2\}$. Intuitively, he may spend some hours a day ensambling products, some time presenting them to potential customers later on, and finally cleaning the store, or keeping records of all costs and receipts. For simplicity, assume that output y_1 (y_2) is only generated from effort e_1 (e_2), and let g_1 (g_2) denote the agent's efficiency at producing output 1 (2, respectively). Each output behaves similarly as in the previous exercise, that is,

$$\begin{aligned} y_1 &= g_1 e_1+\varepsilon_1 \\ y_2 &= g_2 e_2+\varepsilon_2 \end{aligned}$$

Random shocks ε_1 and ε_2, which affect outputs 1 and 2 respectively, follow a bivariate normal distribution $N\left(\mathbf{0},\Sigma\right)$, with expectation 0 for both outputs and variance-covariance matrix of

$$\Sigma=\begin{pmatrix}\sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2\end{pmatrix}$$

The agent earns a wage of $w=s_1y_1+s_2y_2$, where s_1 and s_2 represent his output shares, with $0 \leq s_i \leq 1$ for $i \in \{1, 2\}$. Cost is increasing and convex in both effort levels, where

$$c(e_1, e_2) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2$$

such that his payoff function now becomes

$$U = u(w) - c(e_1, e_2)$$

and his utility function $u(w)$ is the same as that in part (a).

(a) Find the certainty equivalent payment of the agent.

- The agent's expected utility from earning wage w is

$$E_A[u(w)] = 1 - \int_{-\infty}^{+\infty} \exp(-\eta w) f(w) \, dw$$

where $w = s_1 y_1 + s_2 y_2$ following a normal distribution of $\mathcal{N}(\mu_w, \sigma_w^2)$. Specifically, the agent's expected wage, μ_w, is given by

$$\begin{aligned}
\mu_w &= E[w] \\
&= E\left[s_1 \underbrace{(g_1 e_1 + \varepsilon_1)}_{y_1} + s_2 \underbrace{(g_2 e_2 + \varepsilon_2)}_{y_2}\right] \\
&= s_1 g_1 e_1 + s_1 \underbrace{E[\varepsilon_1]}_{0} + s_2 g_2 e_2 + s_2 \underbrace{E[\varepsilon_2]}_{0} \\
&= s_1 g_1 e_1 + s_2 g_2 e_2
\end{aligned}$$

and the variance of his wage, σ_w^2, is given by

$$\begin{aligned}
\sigma_w^2 &= Var(w) \\
&= Var(s_1(g_1 e_1 + \varepsilon_1) + s_2(g_2 e_2 + \varepsilon_2)) \\
&= Var(s_1 \varepsilon_1 + s_2 \varepsilon_2) \\
&= \begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \\
&= s_1^2 \sigma_1^2 + 2 s_1 s_2 \sigma_{12} + s_2^2 \sigma_2^2
\end{aligned}$$

- Since both μ_w and σ_w^2 are scalar terms (as the agent's wage is a linear combination of his output shares), his wage w follows a univariate normal distribution of

$$f(w) = \frac{1}{\sqrt{2\pi}\sigma_w} \exp\left(-\frac{1}{2}\left(\frac{w-\mu_w}{\sigma_w}\right)^2\right)$$

that is the same as part (a), entailing a certainty equivalent payment of

$$\begin{aligned} CE(e) &= \mu_w - \eta\frac{\sigma_w^2}{2} \\ &= \underbrace{s_1 g_1 e_1 + s_2 g_2 e_2}_{\mu_w} - \frac{\eta}{2}\underbrace{\left(s_1^2\sigma_1^2 + 2s_1 s_2\sigma_{12} + s_2^2\sigma_2^2\right)}_{\sigma_w^2} \end{aligned}$$

(b) How is the agent's certainty equivalent affected when he becomes more risk averse?

- Differentiating the certainty equivalent payment with respect to η, we find

$$\frac{\partial CE(e)}{\partial \eta} = -\frac{1}{2}\left(s_1^2\sigma_1^2 + 2s_1 s_2\sigma_{12} + s_2^2\sigma_2^2\right) < 0$$

such that when the agent becomes more risk averse, he demands a lower certainty equivalent payment. Intuitively, as the agent exhibits lower tolerance for output shocks, he is willing to accept a lower sure payment (that is, a higher risk premium) to make him indifferent to receiving volatile wages.

(c) How is the agent's certainty equivalent affected when output shocks become more volatile?

- Differentiating the certainty equivalent payment with respect to σ_1, σ_2, and σ_{12}, we obtain

$$\begin{aligned} \frac{\partial CE(e)}{\partial \sigma_1} &= -\eta s_1^2\sigma_1 < 0 \\ \frac{\partial CE(e)}{\partial \sigma_2} &= -\eta s_2^2\sigma_2 < 0 \end{aligned}$$

such that when output shocks become volatile (that is, σ_1 or σ_2 increases in magnitude), the risk-averse agent demands a lower certainty equivalent payment.

(d) How is the agent's certainty equivalent affected when output shocks become more correlated?

- Differentiating the certainty equivalent payment with respect to σ_{12}, we find

$$\frac{\partial CE(e)}{\partial \sigma_{12}} = -\eta s_1 s_2 < 0$$

Intuitively, when the risks become more positively (or less negatively) correlated (that is, σ_{12} increases in magnitude), the agent demands a lower certainty equivalent payment. We next elaborate on each case:

- First, when outputs exhibit stronger movement in the same direction (positive correlation), the agent is more likely to receive unfavorable shocks in both outputs 1 and 2 given his efforts, such that he is willing to accept a lower sure payment.
- Second, if output shocks exhibits stronger movement in opposite directions (negative correlation), when the agent receives an unfavorable shock that generates a lower output 1, he is more likely to receive a favorable output shock generating a higher output 2.

Overall, his wages are balanced by the opposite effects originating from each output. As a result, he is willing to accept a higher sure payment (that is, a lower risk premium) that makes him indifferent to receiving volatile wages.