Homework #5 (Due on 09/28/2018) - Solution

1. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce one type of output. The price of every unit of input is $w = 8$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem, and solve for its unconditional factor demand $z(8, p)$.

   - The firm chooses the units of input z to solve

   $$\max_{z \geq 0} p\sqrt{z} - 8z$$

   where the first term indicates total revenue, whereas the second reflects total costs. Taking first-order condition with respect to z, we obtain

   $$p\frac{1}{2}z^{-1/2} - 8 \leq 0.$$

   In the case of interior solutions, we can solve for z to find the unconditional factor demand

   $$z(8, p) = \frac{p^2}{256}.$$

   Hence, total output is $q = \sqrt{\frac{p^2}{256}} = \frac{p}{16}$ units.

   (b) Evaluate the profit function at the unconditional factor demand $z(8, p)$. Test for convexity of the profit function in output price p.

   - Inserting $z(8, p) = \frac{p^2}{256}$ into the firm's objective function, we obtain

   $$\pi(p) = p\sqrt{z(8, p)} - 8z(8, p) = \frac{1}{32}p\,(2p - p) = \frac{p^2}{32},$$

   which is convex in output price p.

   (c) Let us now illustrate convexity in output prices by using an alternative approach: (1) evaluate the profit function you found in part (b) at prices $p = 6$, and at $p = 12$. Then, find their convex combination $\alpha\pi(6) + (1 - \alpha)\,\pi(12)$ where $\alpha \in [0, 1]$; (2) evaluate the profit function at the convex combination of the above output prices, that is, $\pi(\alpha 6 + (1 - \alpha)\,12)$. Last, show that the profit function you found in step (1) lies weakly above that found in step (2) for all values of α, that is,

   $$\alpha\pi(6) + (1 - \alpha)\,\pi(12) \geq \pi(\alpha 6 + (1 - \alpha)\,12).$$

- Evaluating the output function at those two output prices, we obtain $\pi(6) = \frac{9}{8}$ and $\pi(12) = \frac{9}{2}$. Hence, their convex combination is

$$\alpha\pi(6) + (1-\alpha)\,\pi(12) = \alpha\frac{9}{8} + (1-\alpha)\,\frac{9}{2} = \frac{9}{8}(4-3\alpha).$$

If, instead, we evaluate the profit function at an output price $p = \alpha 6 + (1-\alpha)\,12$, we obtain

$$\pi\left(\alpha 6 + (1-\alpha)\,12\right) = \frac{9}{8}(4-4\alpha+\alpha^2)$$

Substracting $[\alpha\pi(6) + (1-\alpha)\,\pi(12)] - \pi(\alpha 6 + (1-\alpha)\,12)$, we find

$$\frac{9}{8}(4-3\alpha) - \frac{9}{8}(4-4\alpha+\alpha^2) = \frac{9}{8}\alpha\,(1-\alpha) > 0.$$

which is positive since $\alpha \in [0,1]$.

2. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce units of output q. The price of every unit of input is $w > 0$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem (PMP), and solve for its unconditional factor demand $z(w,p)$.

   - The firm chooses the units of input z to solve

$$\max_{z\geq 0}\; p\sqrt{z} - wz$$

where the first term indicates total revenue, whereas the second reflects total costs. Taking first-order condition with respect to z, we obtain

$$p\frac{1}{2}z^{-1/2} - w \leq 0.$$

In the case of interior solutions, we can solve for z to find the unconditional factor demand

$$z(w,p) = \frac{p^2}{4w^2}.$$

   (b) What is the output level that arises from using the amount of inputs $z(w,p)$? Label this output level $q(w)$.

- Inserting $z(w,p)$ into the firm's production function $\sqrt{z}$, we obtain

$$q(w) = \frac{p}{2w}$$

(c) Set up the firm's cost-minimization problem (CMP), and solve for its conditional factor demand $z(w,q)$ for any output level q. (For now, we write the constraint of the CMP to be $f(z) \geq q$, where the output level q that the firm seeks to reach does not necessarily coincide with that found in part (b), $q(w)$.)

- The firm chooses the units of input z to solve

$$\min_{z \geq 0} \; w \cdot z$$

$$\text{subject to} \quad \sqrt{z} \geq q$$

Setting up the Lagrangian, we obtain

$$L = w \cdot z - \lambda \left(\sqrt{z} - q \right).$$

Taking first-order condition with respect to z, we find that

$$w - \frac{\lambda}{2\sqrt{z}} = 0,$$

and solving for z, we find

$$z = \frac{\lambda^2}{4w^2}.$$

Now, note that the constraint must be binding in equilibrium, so that $\sqrt{z} = q$. Otherwise, the firm could still reduce its total costs and satisfy the output constraint (reaching output target q). Using the binding constraint $\sqrt{z} = q$ into the above result, we obtain that

$$\lambda = 2qw$$

Last, we solve for z, to find the conditional factor demand

$$z(w,q) = q^2$$

(d) Evaluate the conditional factor demand $z(w,q)$ at output level $q = q(w)$, to obtain $z(w, q(w))$. Show that it coincides with the unconditional factor demand $z(w,p)$

found in part (a), that is,

$$z(w, q(w)) = z(w, p).$$

- We find that

  $$z(w, q(w)) = \left(\frac{p}{2w}\right)^2 = \frac{p^2}{4w^2} = z(w, p)$$

  which coincides with the unconditional factor demand $z(w, p)$ found in part (a).

(e) *Shephard's lemma.* Evaluate the CMP's objective function, $w{\cdot}z$, at the conditional factor demand $z(w, q)$, to obtain the cost function, that is, find $c(w, q) = w \cdot z(w, q)$. Differentiate the cost function with respect to w, and show that your result coincides with the conditional factor demand $z(w, q)$.

- The cost function is

  $$c(w, q) = w \cdot z(w, q) = wq^2$$

  Differentiating with respect to input price w, we obtain

  $$\frac{\partial c(w, q)}{\partial w} = q^2$$

  which coincides with the conditional factor demand $z(w, q)$ found in part (c).

(f) *Substitution and output effects.* Let us now consider that the firm faces cheaper wages (lower w). Differentiate the unconditional factor demand $z(w, p)$ found in part (a) with respect to w to find the total effect of this price change.

- Differentiating $z(w, p)$ with respect to input price w, we obtain

  $$\frac{\partial z(w, p)}{\partial w} = -\frac{p^2}{2w^3}$$

  which is negative, thus indicating that higher wages induce the firm to hire fewer workers.

(g) Differentiate the conditional factor demand $z(w, q)$ found in part (c) with respect to w to obtain the substitution effect of this price change.

- Differentiating $z(w, q)$ with respect to input price w, we obtain

  $$\frac{\partial z(w, q)}{\partial w} = 0$$

  In this case, this derivative reflects that, if the firm had to solve the CMP at the new input price (while still reaching the same output target q), it would have to choose same workers.

(h) Compare your results in parts (f) and (g). Which is the output effect of the change in w?

- Comparing $\frac{\partial z(w,p)}{\partial w}$ (which captures the total effect) and $\frac{\partial z(w,q)}{\partial w}$ (which only measures the substitution effect), we find that the output effect is

$$\frac{\partial z(w,p)}{\partial w} - \frac{\partial z(w,q)}{\partial w} = -\frac{p^2}{2w^3}$$

  which is also negative. Hence, as wages increase, the firm chooses to produce fewer units, which ultimately reduces its factor demand (hiring fewer workers).