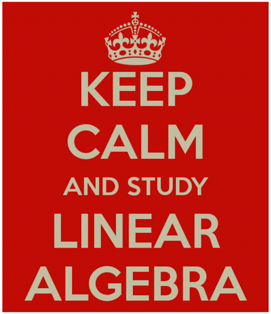
Matrix Algebra

Chapter 8 - S&B



Algebraic operations

- **Matrix:** *The size of a matrix is indicated by the number of its rows and the number of its columns. A matrix with* k *rows and* n *columns is called a* $k \times n$ *matrix. The number in a row* i *and column* j *is called the* (i,j)*th entry, and is often written as* a_{ij}*. Two matrices are equal if they both have the same size and if the corresponding entries in the two matrices are equal.*

Algebraic operations

- **Addition/Substraction:** One can add two matrices of the same size (same number of rows and columns). The result is a new matrix of the same size.

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ . & a_{ij} & . \\ a_{k1} & \dots & a_{kn} \end{pmatrix} + \begin{pmatrix} b_{11} & \dots & b_{1n} \\ . & b_{ij} & . \\ b_{k1} & \dots & b_{kn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ . & a_{ij}+b_{ij} & . \\ a_{k1}+b_{k1} & \dots & a_{kn}+b_{kn} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 3 & 4 & 1 \\ 6 & 7 & 0 \\ -1 & 3 & 8 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 7 \\ 6 & 5 & 1 \\ -1 & 7 & 0 \end{pmatrix} = ?$$

Algebraic operations

- **Scalar Multiplication.** It is the product of the matrix A and the number r, denoted rA, is the matrix created by multiplying each entry of A by r
- $$r\begin{pmatrix} a_{11} & \dots & a_{1n} \\ . & a_{ij} & . \\ a_{k1} & \dots & a_{kn} \end{pmatrix} = \begin{pmatrix} ra_{11} & \dots & ra_{1n} \\ . & ra_{ij} & . \\ ra_{k1} & \dots & ra_{kn} \end{pmatrix}$$

Example:

$$3\begin{pmatrix} 3 & 4 & 1 \\ 6 & 7 & 0 \\ -1 & 3 & 8 \end{pmatrix} = ?$$

- **Matrix Multiplication**.Not all pairs of matrices can be multiplied together, and the order in which matrices are multiplied can matter. We can define the matrix product AB if and only if: number of columns of A = number of rows of B
- For the matrix product to exist, A must be $k \times m$ and B must be $m \times n$. To obtain the (i, j)th entry of AB, multiply the ith row of A and the jth column of B. For example,

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \\ eA + fC & eB + fD \end{pmatrix}$$

- Note that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

- is not defined!

- If A is $k \times m$ and B is $m \times n$, then the product AB will be $k \times n$. The product matrix AB inherits the number of its rows from A and the number of its columns from B.
- the $n \times n$ matrix

$$I = \begin{pmatrix} 1\ 0 \ldots & 0 \\ 0\ 1 & 0 \\ 0\ 0 \ldots & 1 \end{pmatrix}$$

- with $a_{ii} = 1$ for all i and $a_{ij} = 0$ for all $i \neq j$, has the property that for any $m \times n$ matrix A

$$AI = A$$

and for any $n \times l$ matrix B

$$IB = B$$

the matrix I is called *identity matrix* because it is a multiplicative identity for matrices just as the number 1 is for real numbers.

Laws of Matrix Algebra

- *Associate Laws:* $(A+B)+C = A+(B+C)$
- *Commutative Law for Addition:* $A+B = B+A$
- *Distributive Laws:* $A(B+C) = AB+AC$ or $(A+B)C = AC+BC$
- NO *Commutative Law for Multiplication: It is not true that* $AB = BA$ *for matrices, even when both products are defined. That is, even if both products exist, they not be the same size. Check the following example (prove that* $AB \neq BA$*):*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

Transpose

- The transpose of a $k \times n$ matrix A is the $n \times k$ matrix obtained by interchanging the rows and columns of

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

- The following rules are fairly straightforward to verify:
  - $(A+B)^T = A^T + B^T$
  - $(A-B)^T = A^T - B^T$
  - $(A^T)^T = A$
  - $(rA)^T = rA^T$
- Theorem 8.1. Let A be a $k \times m$ matrix and B be an $m \times n$ matrix. Then $(AB)^T = B^T A^T$

System of equations in matrix form

- $Ax = b$

$$\begin{pmatrix} a_{11} & & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & & a_{kn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Special kinds of matrices

- **Square Matrix.** $k = n$ (equal number of rows and columns)
- **Column Matrix.** $n = 1$ (one column)
- **Row Matrix**. $k = 1$ (one row)
- **Diagonal Matrix:** $k = n$ and $a_{ij} = 0$ for all $i \neq j$, that is, a square matrix in which all nondiagonals entries are zero.

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

- **Upper-Triangular Matrix:**$a_{ij} = 0$ if $i > j$, that is, a matrix (usually square) in which all entries below the diagonal are zero.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Special kinds of matrices

- **Lowe-Triangular Matrix:** $a_{ij} = 0$ if $i < j$, that is, a matrix (usually square) in which all entries above the diagonal are zero.

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

- **Symmetric Matrix.** $A^T = A$, that is, $a_{ij} = a_{ji}$ for all i, j. These matrices are necessarily square.

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

- **Idempotent Matrix**. $B \cdot B = B$. i.e., $B = I$ or

$$\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}$$

Special kinds of matrices

- **Permutation Matrix**. A matrix of 0s and 1s in which each row and each column contains exactly one 1.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **Nonsingular Matrix**. A square matrix whose rank equals the number of its rows. When such a matrix arises as a coefficient matrix in a system of linear equations, the system has an unique solution.

Elementary matrices

Recall 3 elementary row operations used to bring a matrix to a row echelon form:

1. interchanging rows,
2. adding a multiple of one row to another, and
3. multiplying a row by a nonzero scalar

These operation can be performed on matrix A by premultiplying A by certain special matrices called *elementary matrices*.

Elementary matrices

- **Theorem:** From the permutation matrix E_{ij} by interchanging the ith and jth rows of the Identity matrix. Left-multiplication of a given matrix A by E_{ij} has the effect of interchanging the ith and jth rows of A.

- **Theorem:** For any $k \times n$ matrix A there exist elementary matrices $E_1, E_2, \ldots, E_m$ such that the matrix product $E_m \cdot E_{m-1} \cdots E_1 \cdot A = U$ where U is in (reduced) row echelon form.

Algebra of Square Matrices

- **Definition.** Let A be a matrix in M_n. The matrix B in M_n is an inverse for A if $AB = BA = I$
  - If B exists then A is invertible
- **Definition.** Let A be an $k \times n$ matrix. The $n \times k$ matrix B is a *right inverse* for A if $AB = I$. the $n \times k$ matrix C is a *left inverse* for A if $CA = I$.
- **Theorem 8.6**. If an $n \times n$ matrix A is invertible, then it is nonsingular, and the unique solution to the system of linear equations $Ax = b$ is $x = A^{-1}b$.