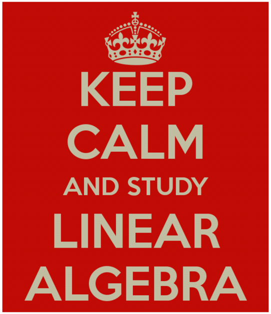
System of Linear Equations

Chapter 7 - S&B



Gaussian and Gauss-Jordan Elimination

- We will study systems of linear equations by describing techniques for solving such systems. The preferred solution technique- *Gaussian elimination*- answers the fundamental questions about a given linear system: **does a solution exist, and if so, how many solutions are there?**
- In general

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

- In this system, the a_{ij}'s and b_i's are given real numbers;
- a_{ij} is the coefficient of the unknown x_j in the ith equation.
- A solution of a system is an n-tuple of real numbers x_1, x_2,. ,x_n, which satisfies each of the m equations.

Gaussian and Gauss-Jordan Elimination

- We are interested in the following three questions:

1. Does a solution exist?
2. How many solutions are there?
3. Is there an efficient algorithm that computes actual solutions?

- There are three ways of solving such systems: (1) Substitution, (2) elimination of variables, and (3) matrix methods.

Substitution

- Solve one equation of system for one variable, say x_n in terms of the other variables in that equation.
- Substitute this expression for x_n into the other $m - 1$ equations.
- The result is a new system of $m - 1$ equations in the $n - 1$unknowns $x_1, ..., x_{n-1}$
- Continue this process by solving one equation in the new system for x_{n-1} and substituting this expression into the other $m - 2$ equations to obtain a system of $m - 2$ equations in the $n - 2$ variables $x_1, ..., x_{n-2}$.

Substitution

- $$\begin{aligned} x_1 &= 0.4x_2 + 0.3x_3 + 130 \\ x_2 &= 0.2x_1 + 0.12x_2 + 0.14x_3 + 74 \\ x_3 &= 0.5x_1 + 0.2x_2 + 0.05x_3 + 95 \end{aligned}$$

Elimination of variables

- $$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ -0.2x_1 + 0.88x_2 - 0.14x_3 &= 74 \\ -0.5x_1 + 0.2x_2 + 0.95x_3 &= 95 \end{aligned}$$
- We will also have to use the method **back substitution**. This procedure is called **Gaussian elimination**.

Elementary Row Operations

- Coefficient Matrix and Augmented Matrix
- Example

$$\begin{aligned} x_1 - 2x_2 &= 8 \\ 3x_1 + x_2 &= 3 \end{aligned}$$

- $$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -2 & 8 \\ 3 & 1 & 3 \end{pmatrix}$$

- Our three elementary equation operations now become elementary row operations:
  - Interchange two rows of a matrix.
  - Change a row by adding to it a multiple of another row, and
  - Multiply each element in a row by the same non-zero number.

Elementary Row Operations

- **Definition.** A row of a matrix is said to have k **leading zeros** if the first k elements of the row are all zeros and the $(k+1)$th element of the row is not zero. With this terminology, a matrix is in **row echelon form** if each row has more leading zeros than the row preceding it.
- Example

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 6 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 5 & 2 \\ 2 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 7 \\ 9 & 0 \\ 0 & 2 \end{pmatrix}$$

Elementary Row Operations

- **Identity Matrix**

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

- Zero Matrix

$$0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Elementary Row Operations

- **Pivot**. It is the first nonzero entry in each row of a matrix in row echelon form
- Let us analyze the following example

$$\left(\begin{array}{ccc|c} 1 & -0.4 & -0.3 & 130 \\ 0 & 1 & -0.25 & 125 \\ 0 & 0 & 1 & 300 \end{array}\right)$$

- **Definition**. A row echelon matrix in which each pivot is a I and in which each column containing a pivot contains no other nonzero entries is said to be in **reduced row echelon form**.

System with many or no solutions

- In general two lines in the plane will be nonparallel and will cross in exactly one point. However, they can be parallel to each other. In this case, they will either coincide or they will never cross. If they coincide; every point on either line is a solution (infinite many solutions).
- Let us analyze the following example

$$\begin{aligned} p_1 + 2p_2 + 3p_3 &= 1 \\ 3p_1 + 2p_2 + p_3 &= 1 \end{aligned}$$

Rank - The Fundamental Criterion

In order to answer the questions related to existence and uniqueness of solutions we need to analyze the rank.

- **Definition.** The rank of a matrix is the number of nonzero rows in its row echelon form.

$$\begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

- **Fact 7.1.** Let A and $\widehat{A}$ be the coefficient matrix and augmented matrix respectively of a system of linear equations. Then

$$\begin{aligned} rank\ A &\leq rank\ \widehat{A} \\ rank\ A &\leq number\ of\ rows\ of\ A \\ rank\ A &\leq number\ of\ columns\ of\ A \end{aligned}$$

- **Fact 7.2**. A system of linear equations with coefficient matrix A and augmented matrix $\widehat{A}$ has a solution if and only if

$$rank\ \widehat{A} = rank\ A$$

- **Fact 7.3**. A linear system of equations must have either no solutions, one solution, or infinitely many solutions.
- **Fact 7.4.** A system with a unique solution must have at least as many equations as unknowns.
- *Homogeneous system* (they have at least one solution)

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots &= . \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

- **Fact 7.6**. A homogeneous system of linear equations which has more unknows than equations must have infinitely many distinct solutions
- **Fact 7.7**. A system of linear equations with coefficient matrix A will have a solution for every choice of RHS $b_1, .., b_n$ if and only if
$$rank\ A = number\ of\ rows\ of\ A$$
- **Fact 7.9**. Any system of linear equations having A as its coefficient matrix will have at most one solution for every choice of RHS b1, .., bm if and only if
$$rank\ A = number\ of\ columns\ of\ A$$
- **Fact 7.10.** A coefficient matrix A is nonsingular, that is, the corresponding linear system has one and only solution for every choice of RHS is
$$number\ of\ rows\ of\ A = number\ of\ columns\ of\ A = rank\ A$$

Rank - The Fundamental Criterion

- **Fact 7.11**. Consider the linear system of equations $Ax = b$
  - If the number of equations $<$ the number of unknowns, then:
    - $Ax = 0$ has infinitely many solutions,
    - for any given b, $Ax = b$ has 0 or infinitely many solutions, and
    - if rank $A =$ number of equations, $Ax = b$ has infinitely many solutions for every RHS b.
  - If the number of equations $=$ the number of unknowns, then:
    - $Ax = 0$ has one or infinitely many solutions.
    - for any given b, $Ax = b$ has 0, 1, or infinitely many solutions, and
    - if rank $A =$ number of unknowns $=$ number of equations, $Ax = b$ has exactly 1 solution for every RHS b.