Recitation #5 (09/21/2018)

1. Consider a representative consumer with utility function

$$u(q_1, q_2) = \ln q_1 + q_2,$$

where q_1 denotes gallons of gas and q_2 is a numeraire representing all other goods. The price of q_2 is therefore normalized to one, $p_2 = 1$, while the price of gas is $p_1(1+t)$, where $t \in [0, 1]$ represents a specific tax per gallon of gas. For simplicity, assume that the consumer's income is $m > 0$.

(a) Find the Walrasian demand for q_1 and q_2, denoting them as q_1^W and q_2^W, and distinguish the case in which $m > 1$ and that when $m \leq 1$.

- The consumer's utility maximization problem (UMP) is selecting q_1 and q_2 to solve

$$\begin{aligned} \max_{q_1, q_2} \quad u(q_1, q_2) &= \ln q_1 + q_2 \\ \text{subject to} \quad p_1(1+t)q_1 + q_2 &= m \end{aligned}$$

Plugging the constraint into the objective function, the UMP can be simplified to a maximization problem with a single choice variable, q_1, as follows

$$\max_{q_1} \; u(q_1, q_2) = \ln q_1 + [m - p_1(1+t)q_1]$$

Taking first-order conditions with respect to q_1, we obtain $\frac{1}{q_1} - p_1(1+t) = 0$. Solving for q_1 yields the Walrasian demand for good 1

$$q_1^W(\mathbf{p}, m; t) = \frac{1}{p_1(1+t)}$$

where $\mathbf{p} \equiv (p_1, p_2) \in \mathbb{R}^2_{++}$ denotes the price vector. Plugging this result into the budget constraint, we find the Walrasian demand for good 2

$$q_2^W(\mathbf{p}, m; t) = m - p_1(1+t)\frac{1}{p_1(1+t)} = m - 1$$

Note that there is a corner solution when $m \leq 1$ in which good 2 is not consumed. In this case, the trucker spends all his income on good 1, i.e., $p_1(1+t)q_1 + 0 = m$ which, solving for q_1 yields $q_1 = \frac{m}{p_1(1+t)}$. We can, hence, summarize the Walrasian demand correspondence as follows

$$\left(q_1^W(\mathbf{p}, m; t), q_2^W(\mathbf{p}, m; t)\right) = \begin{cases} \left(\frac{1}{p_1(1+t)}, m-1\right) & \text{if } m > 1 \text{, and} \\ \left(\frac{m}{p_1(1+t)}, 0\right) & \text{if } m \leq 1 \end{cases}$$

Note that in the interior solution (when $m > 1$) the demand for gas (good 1) does not depend on income m, i.e., its income effect is zero.

(b) Find the associated indirect utility function, $v\left(q_1^W, q_2^W\right)$.

- Plugging the above results into the trucker's utility function, we obtain an indirect utility function $v_1(\mathbf{p}, m; t) = \ln q_1^W + q_2^W$, that is

$$v_1(\mathbf{p}, m; t) = \begin{cases} \ln\left(\frac{1}{p_1(1+t)}\right) + m - 1 & \text{if } m > 1 \text{, and} \\ \ln\left(\frac{m}{p_1(1+t)}\right) & \text{if } m \leq 1 \end{cases}$$

After months of lobbying from consumers' associations, the government is considering implementing either of the following policies: (1) reduce the tax on gas, from t to $t' = t - \alpha$; or (2) maintain the tax at t but give a subsidy of S dollars to the consumer equal to the tax revenue collected by the tax on gas.

(c) Let us first consider that the consumer's income satisfies $m > 1$, i.e., the consumer is relatively rich. Find the consumer's indirect utility function if the government implements the first policy, $v^I\left(q_1^W, q_2^W\right)$, and if the government implements the second policy, $v^{II}\left(q_1^W, q_2^W\right)$. Under which conditions does the consumer prefer the first policy?

- Since $m > 1$ we are at the interior solution. If the *first* policy is implemented, reducing the tax rate to $t' = t - \alpha$, the consumer's indirect utility function becomes

$$v^I\left(q_1^W, q_2^W\right) = \ln\left(\frac{1}{p_1(1+t-\alpha)}\right) + m - 1$$

If, instead, the *second* policy is implemented, then the tax rate is unaltered, but tax revenue

$$R = tp_1 q_1^W = tp_1 \frac{1}{p_1(1+t)} = \frac{t}{1+t}$$

is given to the consumer in the form of a subsidy, yielding an indirect utility function of

$$v^{II}\left(q_1^W, q_2^W\right) = \ln\left(\frac{1}{p_1(1+t)}\right) + m - 1 + \overbrace{\frac{t}{1+t}}^{\text{subsidy}}$$

since we know that any increase in his wealth is only used to increase the amount of q_2 being consumed (the amount of good q_1 does not increase in income).

- We can now compare $v^I\left(q_1^W, q_2^W\right)$ and $v^{II}\left(q_1^W, q_2^W\right)$, obtaining that

$$v^I\left(q_1^W, q_2^W\right) > v^{II}\left(q_1^W, q_2^W\right) \iff \ln\left(\frac{1+t}{1+t-\alpha}\right) > \frac{t}{1+t}$$

which implies that if the tax reduction α is sufficiently high, the consumer prefers the first to the second policy. Indeed, solving for α, we find that $\alpha > \overline{\alpha}$, where $\overline{\alpha} \equiv (1+t)\left(1-e^{-\frac{t}{1+t}}\right)$.

- Figure 1 depicts cutoff $\overline{\alpha}$, for different values of t. Hence, the region of (t, α)-pairs above cutoff $\overline{\alpha}$ describe settings in which the consumer prefers the first policy (tax reduction), while for (t, α)-pairs below cutoff $\overline{\alpha}$ the consumer prefers the second policy (subsidy).

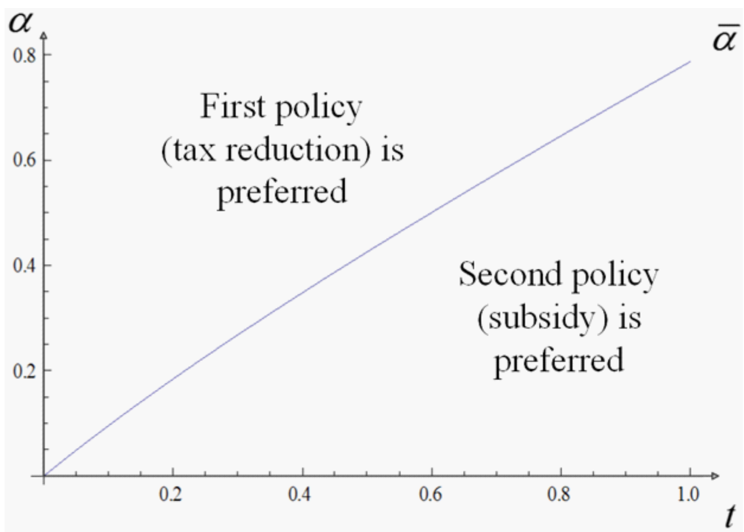

Figure 1. Policy comparison when $m > 1$.

- *Numerical example-I.* If the initial tax on gas is $t = 0.5$, then we obtain that $\ln\left(\frac{1+0.5}{1+0.5-\alpha}\right) > \frac{0.5}{1+0.5}$ holds if the tax reduction, α, satisfies $\alpha > 0.42$. If the initial tax is higher, $t = 0.8$, then we obtain that $\ln\left(\frac{1+0.8}{1+0.8-\alpha}\right) > \frac{0.8}{1+0.8}$ holds only if $\alpha > 0.64$. Intuitively, as the tax rate becomes higher, the consumer needs a larger tax reduction, α, in order to make the first policy preferable.

(d) Let us now consider that the consumer's income satisfies $m \leq 1$, i.e., the consumer is relatively poor. Find the consumer's indirect utility function if the government implements the first policy, $v^I\left(q_1^W, q_2^W\right)$, and if the government implements the second policy, $v^{II}\left(q_1^W, q_2^W\right)$. Under which conditions does the consumer prefer the first policy?

- Since $m \leq 1$, we are at the corner solution, which is $\left(q_1^W, q_2^W\right) = \left(\frac{m}{p_1(1+t)}, 0\right)$. If the *first* policy is implemented, reducing the tax rate to $t' = t - \alpha$, the consumer's indirect utility function becomes

$$v^I\left(q_1^W, q_2^W\right) = \ln\left(\frac{m}{p_1(1+t-\alpha)}\right).$$

If, instead, the *second* policy is implemented, then the tax rate is unaltered, but tax revenue $R = tp_1\frac{m}{p_1(1+t)} = \frac{t}{1+t}m$ is given to the consumer in the form of a subsidy, yielding a total income of $m + \frac{t}{1+t}m = \frac{1+2t}{1+t}m$. Note that $m \leq 1$ does not guarantee that $\frac{1+2t}{1+t}m \leq 1$, so with this new income, the consumer is still at a corner solution if $\frac{1+2t}{1+t}m \leq 1$, or moves to an interior solution if $\frac{1+2t}{1+t}m > 1$, i.e., where $\frac{1+2t}{1+t}m > 1 > m$. Let us analyze each case separately:

- **CASE 1:** When $\frac{1+2t}{1+t}m \leq 1$, Walrasian demands are given by

$$\left(q_1^W(\mathbf{p}, m; t), q_2^W(\mathbf{p}, m; t)\right) = \left(\frac{m(1+2t)}{p_1(1+t)^2}, 0\right)$$

yielding an indirect utility function of

$$v^{II-1^{st}}\left(q_1^W, q_2^W\right) = \ln\left(\frac{m(1+2t)}{p_1(1+t)^2}\right)$$

We can now compare the indirect utility function of the consumer in the first and second policy, i.e., $v^I\left(q_1^W, q_2^W\right)$ and $v^{II-1^{st}}\left(q_1^W, q_2^W\right)$, which implies that the consumer prefers the first policy if

$$\ln\left(\frac{m}{p_1(1+t-\alpha)}\right) > \ln\left(\frac{m(1+2t)}{p_1(1+t)^2}\right) \iff \ln\left(\frac{(1+t)^2}{(1+t-\alpha)(1+2t)}\right) > 0$$

Solving for α, we obtain that for $\alpha > \frac{t(1+t)}{1+2t} \equiv \widetilde{\alpha}$, a tax reduction is preferred over a subsidy.

- **CASE 2:** When $\frac{1+2t}{1+t}m > 1$, Walrasian demands are now both interior, and given by

$$\left(q_1^W(\mathbf{p}, m; t), q_2^W(\mathbf{p}, m; t)\right) = \left(\frac{1}{p_1(1+t)}, \frac{1+2t}{1+t}m - 1\right)$$

yielding an indirect utility function of

$$v^{II-2^{nd}}\left(q_1^W, q_2^W\right) = \ln\left(\frac{1}{p_1(1+t)}\right) + \frac{1+2t}{1+t}m - 1$$

We can now compare the indirect utility function of the consumer in the first and second policy, i.e., $v^I\left(q_1^W, q_2^W\right)$ and $v^{II-2^{nd}}\left(q_1^W, q_2^W\right)$, which implies that the consumer prefers the first policy if

$$\ln\left(\frac{m}{p_1\left(1+t-\alpha\right)}\right) > \ln\left(\frac{1}{p_1(1+t)}\right) + \frac{1+2t}{1+t}m - 1$$

or rearranging,

$$\ln\left(\frac{m(1+t)}{1+t-\alpha}\right) > \frac{1+2t}{1+t}m - 1$$

Solving for α, we find that $\alpha > 1+t-(1+t)me^{1+m\left(\frac{t}{1+t}-2\right)} \equiv \widehat{\alpha}$.

- *Numerical example-II.* If the initial tax rate is $t = 0.5$ and the consumer's income is $m = 0.8$, then the above condition holds as long as $\alpha > \widehat{\alpha} = 0.37$. Similarly as in part (c) of the exercise, if the tax rate is higher, $t = 0.8$, then the above condition holds as long as $\alpha > \widehat{\alpha} = 0.56$, suggesting that, as the tax rate increases the consumer only prefers the first policy if the size of the tax reduction, α, becomes sufficiently large.
- *Summary.* Finally, note that if the tax reduction is sufficiently larger, i.e., α exceeds all the cutoffs identified in the previous section of the exercise $\alpha > \max\{\overline{\alpha}, \widetilde{\alpha}, \widehat{\alpha}\}$, then the consumer unambiguously prefers a tax reduction than a subsidy regardless of his actual income level, m. If, instead, the tax reduction is extremely low, i.e., α falls below all cutoffs $\alpha < \min\{\overline{\alpha}, \widetilde{\alpha}, \widehat{\alpha}\}$, the consumer prefers the subsidy policy independently of his income level, m. If the tax reduction is, however, intermediate, i.e., $\min\{\overline{\alpha}, \widetilde{\alpha}, \widehat{\alpha}\} < \alpha < \max\{\overline{\alpha}, \widetilde{\alpha}, \widehat{\alpha}\}$, then the consumer's preference for one particular policy depends on his income level.

2. Consider a Cobb-Douglas production function $f : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, given by $f(z) = 2^{3/4} z_1^{1/4} z_2^{1/4}$, where $z_1 \geqslant 0$ and $z_2 \geqslant 0$ denote inputs in the production process.

   (a) Check if the production function has nonincreasing, nondecreasing, or constant returns to scale.

   - *Nonincreasing returns to scale.* If the production function satisfies nonincreasing returns to scale, for all inputs $z \in \mathbb{R}_+^2$ and for all $\alpha > 1$, we must have $\alpha f(z) \geq f(\alpha z)$. Intuitively, increasing all inputs by a common factor α, yields a less-than-proportional increase in output, $f(\alpha z)$, i.e., $\alpha f(z) \geq f(\alpha z)$. In this exercise, this condition implies

$$\alpha \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \geq 2^{3/4} \left(\alpha z_1\right)^{1/4} \left(\alpha z_2\right)^{1/4}$$

Simplifying the right-hand side, we obtain

$$\alpha \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \geq \alpha^{1/2} \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \Leftrightarrow \alpha \geq \alpha^{\frac{1}{2}}$$

which is satisfied for all $\alpha > 1$. Hence, this production function satisfies nonincreasing returns to scale.

- *Nondecreasing returns to scale.* (Since this production function exhibits nonincreasing returns to scale, and such property holds strictly, we can actually anticipate that it will not satisfy nondecreasing returns to scale. However, and as a practice, we go over these properties nevertheless.) If the production function satisfies nondecreasing returns to scale, for all inputs $z \in \mathbb{R}_+^2$ and for all $\alpha > 1$, we must have $\alpha f(z) \leq f(\alpha z)$. In this case, a common increase in all inputs by a common factor α, yields a more-than-proportional increase in output, $f(\alpha z)$, i.e., $\alpha f(z) \leq f(\alpha z)$. In this exercise, this condition implies

$$\alpha \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \leq \alpha^{1/2} \left(2^{3/4} z_1^{1/4} z_2^{1/4}\right) \Leftrightarrow \alpha \leq \alpha^{\frac{1}{2}}$$

  and this inequality cannot hold for any $\alpha > 1$. Then, this production function cannot exhibit nondecreasing returns to scale.

- *Constant returns to scale.* If the production function satisfies constant returns to scale, it must satisfy nonincreasing and nondecreasing returns to scale. Since this production function does not satisfy both, it cannot exhibit constant returns to scale. In particular, when a production function exhibits constant returns to scale, a common increase of all inputs by a common factor $\alpha > 1$, yields a proportional increase in output, $f(\alpha z)$, i.e., $\alpha f(z) = f(\alpha z)$.

- *Remark*: This production function is a standard Cobb-Douglas production function $f(z) = A z_1^{\alpha} z_2^{\beta}$. It is good to remember that when $\alpha + \beta \leq 1$ the production function has nonincreasing returns to scale, when $\alpha + \beta \geq 1$ it has nondecreasing returns to scale, and when $\alpha + \beta = 1$ it exhibits constant returns to scale.

(b) Let $w \in \mathbb{R}_{++}^2$ denote the vector of input prices and $p > 0$ the output price. Determine for each output level $q \geqslant 0$ the cost function $c(w, q)$ and the conditional factor demand $z(w,q)$.

- We first need to find the conditional factor demand (solving the cost minimization problem, CMP, of the firm), and afterwards we can compute the firm's cost function, as the value function emerging from the CMP.

$$\text{CMP} : \min_{z \geq 0} \; w \cdot z$$

$$\begin{aligned} \text{subject to} \quad 2^{3/4} z_1^{1/4} z_2^{1/4} &\geq q, \\ z &\geq 0 \end{aligned}$$

First, note that $z = 0$ can be ruled out. Indeed, from the production function we know that output would be zero when either of the inputs are zero, i.e., $z_1 = 0$ or $z_2 = 0$. Secondly, the production function constraint $2^{3/4} z_1^{1/4} z_2^{1/4} \geq q$ must be binding since the production function is strictly increasing in both inputs and inputs are costly (they are not free since the vector of input prices $w \in \mathbb{R}^2_{++}$ is strictly positive in all components).[1] Thus, we can solve for z_1 in this constraint, finding

$$2^{3/4} z_1^{1/4} z_2^{1/4} = q \Longleftrightarrow z_1 = \frac{1}{8} \frac{q^4}{z_2}.$$

We can now substitute this result into the previous cost minimization problem, thus reducing the number of choice variables to only one, z_2, as follows

$$\min_{z_2} w_1 \cdot z_1 + w_2 \cdot z_2 = w_1 \cdot \left(\frac{1}{8} \frac{q^4}{z_2} \right) + w_2 \cdot z_2$$

The first order condition with respect to z_2 is

$$-w_1 \left(\frac{1}{8} \frac{q^4}{z_2^2} \right) + w_2 = 0$$

and solving for z_2, yields

$$z_2 = \frac{1}{2} q^2 \sqrt{\frac{w_1}{2w_2}}$$

[1] Alternatively, one can set up the Lagrangian of the firm's profit maximization problem (PMP) using λ as the Lagrange multiplier of constraint $2^{3/4} z_1^{1/4} z_2^{1/4} \geq q$, then take first-order conditions with respect to inputs z_1 and z_2, and obtain that $\lambda > 0$, which implies that the above constraint holds with equality.

Substituting z_2 into the expression for z_1 we found above, we obtain

$$z_1 = \frac{1}{8}\frac{q^4}{z_2} \Longrightarrow z_1 = \frac{1}{8}\frac{q^4}{\left(\frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}}\right)} = \frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}}$$

Therefore, the conditional factor demand is

$$z(w,q) = \left(\frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}}, \frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}}\right)$$

As a consequence, the cost function (i.e., the minimal cost that the firm must incur in order to attain output level of q) is

$$\begin{aligned} c(w,q) &= w_1 \cdot z_1(w,q) + w_2 \cdot z_2(w,q) \\ &= w_1\frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}} + w_2\frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}} \\ &= \frac{1}{2}q^2\sqrt{2w_1w_2} \end{aligned}$$

which can be interpreted as the value function of the CMP, since we evaluated the objective function at the arguments that solved the CMP.

(c) Verify Shephard's lemma.

- Let us first recall Shephard's lemma: If the production set is *closed* and satisfies the *free-disposal* property, and the conditional factor demand $z(\overline{w},q)$ consists of a single point $\overline{z}$, then the cost function $c(w,q)$ is differentiable with respect to w at $\overline{w}$, and this derivative is

$$\frac{\partial c(\overline{w},q)}{\partial w_l} = \overline{z}_l$$

Hence, in order to verify Shephard's lemma, we must first check that the production set Y is closed, and that it satisfies the free disposal property.

- *Closedness.* The production set associated with the production function is given by

$$Y = \left\{(-z,q) \in \mathbb{R}^3 : q \leq f(z) \text{ and } z \in \mathbb{R}^2_+\right\}$$

and for convenience, we can rewrite this set as

$$Y = \left\{y \in \mathbb{R}^3 : y_1 \leq 0\right\} \cap \left\{y \in \mathbb{R}^3 : y_2 \leq 0\right\} \cap \left\{y \in \mathbb{R}^3 : y_3 \leq f(-y_1,-y_2)\right\}$$

which is the intersection of three closed sets (the first two representing inputs, and the third representing output), and as a consequence it is closed. [Recall that the intersection of finitely many closed sets is also closed.]

- *Free-disposal.* Consider two input-output pairs that belong to production set Y, $(-z, q) \in Y$ and $(-z', q')$, where $(-z', q') \leq (-z, q)$, as depicted in the figure below. This means that the second pair either uses more inputs as the first pair (producing the same output) or uses the same amount of inputs (but produces a smaller output). That is, either: (1) $z_1' \geq z_1$ or $z_2' \geq z_2$ but producing the same output $q' = q$; or (2) $z_1' = z_1$ and $z_2' = z_2$ but producing less output $q' \leq q$. In order to show that the free-disposal property is satisfied, we must show that $(-z', q')$ also belongs to the production set Y. Since the production function $f(\cdot)$ is weakly increasing in both inputs z_1 and z_2, we find that

$$q' \leq q \leq f(z) \leq f(z')$$

as illustrated in figure 1. That is, $(-z', q')$ also belongs to the production set Y.

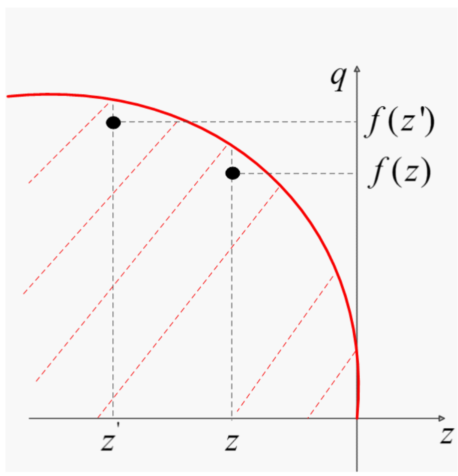

Figure 1. Production function $f(z)$ satisfies free disposal.

- Hence, the production set Y is closed and satisfies free-disposal, implying that all conditions for Shephard's lemma hold.[2] We can thus determine the conditional factor demand function for input 1, $z_1(w, q)$, by differentiating the cost function, $c(w, q)$, with respect to the price of input 1, as follows

$$\frac{\partial c(w,q)}{\partial w_1} = \frac{\partial \left(\frac{1}{2}q^2\sqrt{2w_1w_2}\right)}{\partial w_1} = \frac{1}{2}q^2\sqrt{\frac{w_2}{2w_1}}$$

[2]Note that there is an additional condition, which states that conditional factor demand correspondences consist of a single point (they are functions); and this was clearly satisfied in our exercise. For a given input price vector $w = (w_1, w_2)$ and output q, the function $z(w, q)$ yields a real number for the input usage of z_1 and another for z_2.

and similarly for the conditional factor demand of input 2, $z_2(w, q)$,

$$\frac{\partial c(w,q)}{\partial w_2} = \frac{\partial \left(\frac{1}{2}q^2\sqrt{2w_1w_2}\right)}{\partial w_2} = \frac{1}{2}q^2\sqrt{\frac{w_1}{2w_2}}$$

(d) Determine the profit function $\pi(p, w)$.

- To determine the profit function, we can solve the profit maximization problem using the cost function,

$$\max_{q\geq 0} \; pq - \frac{1}{2}q^2\sqrt{2w_1w_2}$$

Taking first-order conditions with respect to q yields

$$p - q^*\sqrt{2w_1w_2} \leq 0$$

which holds with equality in interior solutions, $q^* > 0$. In the case of interior solutions, we can solve for q^* to obtain the following profit-maximizing output

$$q^* = \frac{p}{\sqrt{2w_1w_2}}.$$

And the profit arising from producing this output level is

$$\begin{aligned}\pi(p,w) &= pq^* - \frac{1}{2}(q^*)^2\sqrt{2w_1w_2} \\ &= \frac{p^2}{2\sqrt{2w_1w_2}}\end{aligned}$$

Again we can see that since $p > 0$ and $w > 0$, the profit from producing q^* is positive for all parameter values. It is therefore never optimal to remain inactive, i.e., set $q^* = 0$ (which gives zero profits).

- *Sufficiency*: Let us now check second order conditions. The above PMP is striclty concave, and thus the output level q^* that we found is profit maximizing, if the cost function is convex in q, which holds in this case since

$$\frac{\partial c(w,q)}{\partial q} = q\sqrt{2w_1w_2} \;\text{ and }\; \frac{\partial^2 c(w,q)}{\partial q^2} = \sqrt{2w_1w_2} > 0$$

for all $w_1, w_2 > 0$.