Solution Homework 1 - EconS 501

1. **[Checking properties of preference relations-I].** Moana and Maui need to find the magical fish hook. Maui lost this weapon after stealing the heart of Te Fiti and his subsequent battle with the lava demon Te Kā. The fish hook was lost in sea, and eventually was found by Maui's arch-rival Tamatoa, who placed the fish hook on his shell as a prize. In order to find the hook they need to combine two techniques of navigation, that is, intense observation of (1) the celestial bodies in the sky (technique x) and (2) the swells of the water (technique y). Maui is an expert in the art of navigation, and he weakly prefers a combination of technique x and y that contains more of observation of the sky, i.e., $(x_1, y_1) \succsim (x_2, y_2)$ if and only if $x_1 \geq x_2 + 1$. For this preference relation in the use of navigation techniques (1 and 2): describe the upper contour set, the lower contour set, the indifference set of bundle $(3, 2)$, and interpret them. Then check whether this preference relation is rational (by separately examining whether they are complete and transitive), monotone, and convex.

a. Bundle (x_1, y_1) is weakly preferred to (x_2, y_2), i.e., $(x_1, y_1) \succsim (x_2, y_2)$ if and only if $x_1 \geqslant x_2 + 1$.

(a)
- Let us first build some intuition on this preference relation. First, note that an individual prefers a bundle x to another bundle y if and only if the first component of bundle x, x_1, contains at least one unit more than the first component of bundle y, i.e., $x_1 \geqslant y_1 + 1$. For instance, (3, 2) is preferred to (1, 2) since $x_1 = 3$ and $x_2 = 1$, thus implying $3 \geqslant 1 + 1 = 2$. Importantly, the individual ignores the content of the second component when comparing two bundles. Let us next describe the upper contour, lower contour, and indifference set of a given bundle, such as (3,2). The upper contour set of this bundle is given by

$$UCS(3, 2) = \{(x_1, y_1) \succsim (3, 2) \Longleftrightarrow x_1 \geq 3 + 1\} = \{(x_1, x_2) : x_1 \geq 4\}$$

while the lower contour set is defined as

$$LCS(3, 2) = \{(3, 2) \succsim (x_1, y_1) \Longleftrightarrow 3 \geq x_1 + 1\} = \{(x_1, x_2) : x_1 \leq 2\}$$

Finally, the consumer is indifferent between bundle (3,2) and the set of bundles where

$$IND(3, 2) = \{(x_1, y_1) \sim (3, 2) \Longleftrightarrow \varnothing\}$$

- *Completeness.* For this property to hold, we need that, for any pair of bundles

(x_1, y_1) and (x_2, y_2), either $(x_1, y_1) \succsim (x_2, y_2)$ or $(x_2, y_2) \succsim (x_1, y_1)$, or both (i.e., $(x_1, y_1) \sim (x_2, y_2)$). Since for this preference relation the indiferent set is empty then it is not complete.

- *Transitivity.* We need to show that, for any three bundles (x_1, y_1), (x_2, y_2) and (x_3, y_3) such that

$$(x_1, y_1) \succsim (x_2, y_2) \text{ and } (x_2, y_2) \succsim (x_3, y_3), \text{ then } (x_1, y_1) \succsim (x_3, y_3)$$

This property holds for this preference relation. In order to show this result, notice that a bundle (x_1, y_1) is preferred to another bundle (x_2, y_2) if its first component, x_1, is larger than that of the other bundle, x_2, by more than one unit, i.e., condition $x_1 \geq x_2 + 1$ is equivalent to $1 \leq x_1 - x_2$. A similar argument can be extended to the comparison between two bundles (x_2, y_2) and (x_3, y_3), where the former is preferred to the latter if and only if the distance between their first components is greater than one, i.e., $1 \leq x_2 - x_3$. Hence, for bundle (x_1, y_1) to be preferred to (x_3, y_3), i.e., $(x_1, y_1) \succsim (x_3, y_3)$, we need that the distance between their first components is greater than one, i.e., $1 \leq x_1 - x_3$; as we next show with an example.
Consider the following three bundles (notice that the second component of every bundle is inconsequential, since the preference ordering only relies on a comparison of the first component of every vector):

$$\begin{aligned} (x_1, y_1) &= (6, 4) \\ (x_2, y_2) &= (5, 1) \\ (x_3, y_3) &= (4, 2) \end{aligned}$$

First, note that $(x_1, y_1) \succsim (x_2, y_2)$ since the difference in their first component is greater (or equal) to one unit, $x_1 \geq x_2 + 1$ (i.e., $6 \geq 5 + 1$). Additionally, $(x_2, y_2) \succsim (x_3, y_3)$ is also satisfied since $x_2 \geq x_3 + 1$ (i.e., $5 \geq 4 + 1$). Therefore, $(x_1, y_1) \succsim (x_3, y_3)$ since the difference between x_1 and x_3 is larger than one unit, $x_1 \geq x_3 + 1$. Hence, this preference relation satisfies transitivity.

- *Monotonicity.* This property is satisfied for this preference relation. In particular, increasing the amount of good 1 yields a new bundle $(x_1 + \varepsilon, y_1)$ that is weakly preferred to the original bundle (x_1, y_1), i.e., the comparison of their first component yields $x_1 + \varepsilon \geq x_1 + 1$, which holds iff $\varepsilon \geq 1$. Similarly, increasing the amount of the second component produces a new bundle $(x_1, y_1 + \varepsilon)$ which is weakly preferred to the original bundle (x_1, y_1). Recall that this individual compares bundles by evaluating the first component alone. Since in this case the amount of the first component is unaffected, then he is indifferent between bundle (x_1, y_1) and $(x_1, y_1 + \varepsilon)$; an indifference that is allowed by the definition of monotonicity. Hence, the preference relation does not satisfies monotonicity.

- *Convexity.* This property implies that the upper contour set must be convex, that is, if bundle (x_1, y_1) is weakly preferred to (x_2, y_2), $(x_1, y_1) \succsim (x_2, y_2)$, then the convex combination of these two bundles is also weakly preferred to (x_2, y_2),

$$\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \succsim (x_2, y_2) \text{ for any } \lambda \in [0, 1]$$

In this case, $(x_1, y_1) \succsim (x_2, y_2)$ implies that $x_1 \geq x_2 + 1$; whereas $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \succsim (x_2, y_2)$ implies

$$\lambda x_1 + (1 - \lambda) x_2 \geq x_2 + 1$$

which simplifies to $\lambda x_1 \geq \lambda x_2 + 1$. However, the premise from $(x_1, y_1) \succsim (x_2, y_2)$, i.e., $x_1 \geq x_2 + 1$, entails that $\lambda x_1 \geq \lambda x_2 + 1$ must also hold. (To see that, note that $x_1 \geq x_2 + 1$ can be written as $(x_1 - x_2) - 1 \geq 0$, while $\lambda x_1 \geq \lambda x_2 + 1$ can be expressed as $\lambda(x_1 - x_2) - 1 \geq 0$, where $\lambda(x_1 - x_2) - 1 \leq (x_1 - x_2) - 1$ since $\lambda \in [0, 1]$.) Therefore, $(x_1 - x_2) - 1 \geq 0$ is not a sufficient condition for $\lambda(x_1 - x_2) - 1 \geq 0$. We also need a more restrictive condition $(x_1 - x_2) \geq 2$. Hence, this preference relation is not convex, since $(x_1 - x_2) \geq 1$. Example, consider (3,2) and (2,2), in this case $(x_1, y_1) \succsim (x_2, y_2)$, however, $0.5 \times 3 + (1 - 0.5) 2 \ngeq 2 + 1$.

2. **[Checking properties of preference relations-II].** Consider the following preference relation defined in $X = \mathbb{R}^2_+$. A bundle (x_1, x_2) is weakly preferred to another bundle (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3y_1 + 2y_2, 2y_1 + 3y_2\}$$

(a) For any given bundle (y_1, y_2), draw the upper contour set, the lower contour set, and the indifference set of this preference relation. Interpret.

- Take a bundle $(2, 1)$. Then,

$$\min\{3 * 2 + 2 * 1, 2 * 2 + 3 * 1\} = \min\{8, 7\} = 7.$$

The upper contour set of this bundle is given by

$$\begin{aligned} UCS(2,1) &= \{(x_1, x_2) \succsim (2,1)\} \\ &= \{\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq 7 \equiv \min\{8, 7\}\} \end{aligned}$$

which is graphically represented by all those bundles in $\mathbb{R}^2_+$ which are strictly above *both* lines $3x_1 + 2x_2 = 7$ and $2x_1 + 3x_2 = 7$. That is, for all (x_1, x_2) strictly above both lines

$$x_2 = \frac{7}{2} - \frac{3}{2}x_1 \text{ and } x_2 = \frac{7}{3} - \frac{2}{3}x_1.$$

(See figure 1.9, which depicts these two lines and shades the set of bundles lying weakly above both lines.)

- On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(2,1) &= \{(2,1) \succsim (x_1, x_2)\} \\ &= \{7 \geq \min\{3x_1 + 2x_2, 2x_1 + 3x_2\}\}, \end{aligned}$$

which is graphically represented by all bundles (x_1, x_2) weakly below the maximum of the lines described above. For instance, bundle $(y_1, y_2) = (2.5, 0)$, which lies on the horizontal axis and between both lines' horizontal intercept, implies

$$\min\{3 \cdot 2.5 + 2 \cdot 0, 2 \cdot 2.5 + 3 \cdot 0\} = \min\{7.5, 5\} = 5$$

thus implying that this consumer prefers bundle $(x_1, x_2) = (2, 1)$ than $(y_1, y_2) = (2.5, 0)$. A similar argument applies to all other bundles lying above $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ and below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$, where bundle $(2.5, 0)$ also belongs; see the triangle that both lines form at the right-hand side of the figure. Similarly, bundles such as $(0, 2.5)$ yield

$$\min\{3 \cdot 0 + 2 \cdot 2.5, 2 \cdot 0 + 3 \cdot 2.5\} = \min\{5, 7.5\} = 5,$$

which implies that the consumer also prefers bundle $(2, 1)$ to $(0, 2.5)$. An analogous argument applies to all bundles above line $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ but below

$x_2 = \frac{7}{3} - \frac{2}{3}x_1$ in the triangle at the left-hand side of figure 2.1.

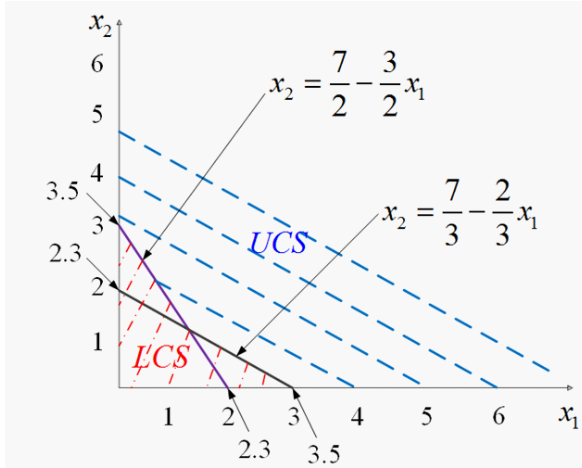

Figure 2.1. UCS and LCS of bundle (2,1).

Finally, those bundles for which the UCS and LCS overlap are those in IND of bundle (2,1).

(b) Check if this preference relation satisfies: (i) completeness, (ii) transitivity, and (iii) weak convexity.

- *Completeness.* First, note that both of the elements in the min$\{\cdot\}$ operator are real numbers, i.e., $(3x_1 + 2x_2) \in \mathbb{R}_+$ and $(2x_1 + 3x_2) \in \mathbb{R}_+$, thus implying that the minimum

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} = a$$

exists and it is also a real number, $a \in \mathbb{R}_+$. Similarly, the minimum

$$\min\{3y_1 + 2y_2, 2y_1 + 3y_2\} = b$$

exists and is a real number, $b \in \mathbb{R}_+$. Therefore, we can easily compare a and b, obtaining that either $a \geq b$, which implies $(x_1, x_2) \succsim (y_1, y_2)$; or $a \leq b$, which implies $(y_1, y_2) \succsim (x_1, x_2)$, or both, $a = b$, which entails $(x_1, x_2) \sim (y_1, y_2)$. Hence, the preference relation is complete.

- *Transitivity.* We need to show that, for any three bundles (x_1, x_2), (y_1, y_2) and (z_1, z_2) such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ implies

$$a \equiv \min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \equiv b$$

and $(y_1, y_2) \succsim (z_1, z_2)$ implies that

$$b \equiv \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\} \equiv c$$

Combining both conditions we have that $a \geq b \geq c$, which implies that $a \geq c$. Hence, we have that

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} \geq \min\{3z_1 + 2z_2, 2z_1 + 3z_2\}$$

and thus $(x_1, x_2) \succsim (z_1, z_2)$, implying that this preference relation is transitive.

- *Weak Convexity.* This property implies that the upper contour set must be convex. That is, if bundle (x_1, x_2) is weakly preferred to (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$, then the convex combination of these two bundles is also weakly preferred to (y_1, y_2),

$$\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2) \text{ for any } \lambda \in [0, 1]$$

For compactness, let $a \equiv 3x_1 + 2x_2$, $b \equiv 2x_1 + 3x_2$, $c \equiv 3y_1 + 2y_2$ and $d \equiv 2y_1 + 3y_2$. Hence, the property that $(x_1, x_2) \succsim (y_1, y_2)$ implies $\min\{a, b\} \geq \min\{c, d\}$. We therefore need to show that

$$\min\{\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d\} \geq \min\{c, d\}$$

1. *First case*: $\min\{a, b\} = a$, $\min\{c, d\} = c$ and $a \geq c$. Therefore,

$$\min\{\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d\} = \lambda a + (1 - \lambda)c$$

and $\lambda a + (1 - \lambda)c > \min\{c, d\} = c$. For this case, convexity is satisfied.

2. *Second case*: $\min\{a, b\} = a$, $\min\{c, d\} = d$ and $a \geq d$. Hence, $a > b$ and $c > d$, implying that

$$\min\{\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d\} = \lambda a + (1 - \lambda)d$$

and $\lambda a + (1 - \lambda)d \geq \min\{c, d\} = d$ given that $a \geq d$. For this case, convexity is satisfied as well. An analogous argument applies in the other two cases, in which $\min\{a,b\} = b$ and $\min\{c,d\} = c$, and in which $\min\{a,b\} = b$ but $\min\{c,d\} = d$.

3. [**Monotonicity and Strong monotonicity.**] Explain monotonicity and strong monotonicity in preference relations, and compare them. Provide an example where a bundle x is (strictly) preferred to bundle y when preferences satisfy strong monotonicity, but x is not necessarily preferred to y under monotonicity.

   - *Monotonicity* states that increasing the amount of some commodities cannot hurt, and increasing the amount of all commodities is strictly preferred. Formally, if we take bundle $y \in \mathbb{R}^L$ and weakly increase all k components, so that we generate a new bundle $x \in \mathbb{R}^L$ satisfying $x_k \geq y_k$ for all k, then an individual with monotonic preferences would prefer the newly created bundle to the original bundle, i.e., $x \succsim y$. (Note that this implies that at least one component of the bundle has been strictly increased while the remaining components can be left unaffected.) In addition, if we strictly increase the amount of all components in bundle y, this individual would strictly prefer the new bundle, i.e., if $x_k > y_k$ for all k, then $x \succ y$.
   - *Strong monotonicity.* On the other hand, strong monotonicity states that the consumer is strictly better off with additional amounts of any commodity. That is, if we strictly increase the amount of at least one commodity, the consumer strictly prefers the newly created bundle x to his original bundle y. That is, if $x_k \geq y_k$ for all good k and $x \neq y$, then $x \succ y$. (Note that this implies that $x_j > y_j$ for at least one commodity j, since otherwise both bundles would coincide.)
   - *Comparison.* Then, a consumer's preference relation can satisfy monotonicity (if additional amounts of one of his commodity do not harm his utility), but does not need to satisfy strong monotonicity (since for that to occur, he would need to become strictly better off as a consequence of the additional amounts in one of his commodities). However, if a consumer's preferences satisfy strong monotonicity, they must also satisfy monotonicity. That is why strong monotonicity is a more restrictive ("stronger") assumption on preferences than monotonicity.
   - *Example*: Consider bundles $x = (1,2)$ and $y = (1,1)$. If preferences satisfy strong monotonicity, $x \succ y$ since the second component in bundle x is higher than the corresponding component in y, i.e., $x_j \geq y_j$ for some good j. However, if preferences only satisfy monotonicity, we cannot state that $x \succ y$ (strictly), since $x_k > y_k$ does not hold for all k commodities.

4. **WARP and Distributive choice rules.** Let $(\mathcal{B}, C(\cdot))$ be a choice structure where $\mathcal{B}$ includes all non-empty subsets of consumption bundles X, i.e., $C(B) \neq \varnothing$ for all sets $B \in \mathcal{B}$. We define the choice rule $C(\cdot)$ to be *distributive* if, for any two sets B and B' in $\mathcal{B}$,

$$C(B) \cap C(B') \neq \varnothing \text{ implies that } C(B) \cap C(B') = C(B \cap B')$$

In words, the elements that the individual decision maker selects both when facing set B and when facing set B', $C(B) \cap C(B')$, coincide with the elements that he would select when confronted with the elements that belong to both sets $B \cap B'$, i.e., $C(B \cap B')$. Show that, if choice rule $C(\cdot)$ is *distributive*, then choice structure $(\mathcal{B}, C(\cdot))$ does not necessarily satisfy the weak axiom of revealed preference. (A counterexample suffices.)

- One possible counterexample is with the consumption set $X = \{x, y, z\}$ and family of budget sets

$$\mathcal{B} = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$$

Let the choice rule $C(\cdot)$ be given by

$$C\{x\} = \{x\}, C\{y\} = \{y\} \text{ and } C\{z\} = \{z\},$$

when facing a single available element,

$$C\{x, y\} = \{y\}, C\{x, z\} = \{x\}, C\{y, z\} = \{y\}$$

when facing two available elements, and

$$C\{x, y, z\} = \{x\}$$

when facing all three elements. First, note that this choice rule is distributive. In particular, the next list considers all possible pairs of budget sets. Specifically, on the left-hand side, the list describes the elements that the decision maker would select both when confronted with one of the budgets sets, B, and with the other budget, B', i.e., it provides the intersection $C(B) \cap C(B')$. On the right-hand side, it reflects the elements that the decision maker would choose when he faces a choice between the common elements of budget sets B and B', i.e., it reports the choice $C(B \cap B')$. As the list confirms, both approaches lead to the same choices from this individual, thus implying that his choice rule is distributive.

$$
\begin{aligned}
C(\{x\}) \cap C(\{x,z\}) &= \{x\} \cap \{x\} = \{x\} = C(\{x\}), \\
C(\{x\}) \cap C(\{x,y,z\}) &= \{x\} \cap \{x\} = \{x\} = C(\{x\}), \\
C(\{x,z\}) \cap C(\{x,y,z\}) &= \{x\} \cap \{x\} = \{x\} = C(\{x,z\}), \\
C(\{y\}) \cap C(\{x,y\}) &= \{y\} \cap \{y\} = \{y\} = C(\{y\}), \\
C(\{y\}) \cap C(\{y,z\}) &= \{y\} \cap \{y\} = \{y\} = C(\{y\}), \\
C(\{x,y\}) \cap C(\{y,z\}) &= \{y\} \cap \{y\} = \{y\} = C(\{y\}).
\end{aligned}
$$

However, note that the weak axiom is not satisfied. In particular, while x and y both belong to $\{x,y\}$ and to $\{x,y,z\}$, this individual selects $C\{x,y\} = \{y\}$ (and does not select x) but changes his choice to x (and not y) when his set of available options expands to include z, i.e., $C\{x,y,z\} = \{x\}$. Thus, the weak axiom fails.[1]

[1]Note that, since the intersection of the chosen sets in $C(\{x,y,z\}) = \{x\}$ and $C(\{x,y\}) = \{y\}$ is empty, i.e., $\{x\} \cap \{y\} = \emptyset$, we cannot apply the definition of distributive rules for this specific case. Nonetheless, since in the above list we found that, for all B and B', the elements selected in $C(B) \cap C(B')$ coincide with those in $C(B \cap B')$, then we can claim that the choice rule is distributive.