Homework #2 (Due Wednesday September 12th, 2018)

1. Consider a consumer with utility function $u(x_1, x_2, x_3) = x_1 x_2 x_3$, and income w.

   (a) Set up the consumer's utility maximization problem and find the Walrasian demands for each good.

   - The consumer solves

$$\max_{x_1,x_2,x_3} \quad x_1x_2x_3$$
$$\text{s.t.} \quad p_1x_1 + p_2x_2 + p_3x_3 \leq w$$

Setting up the Lagrangian, we write

$$L = x_1x_2x_3 + \lambda(w - p_1x_1 - p_2x_2 - p_3x_3)$$

which yields the first-order conditions

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= x_2x_3 - \lambda p_1 = 0 \\
\frac{\partial L}{\partial x_2} &= x_1x_3 - \lambda p_2 = 0 \\
\frac{\partial L}{\partial x_3} &= x_1x_2 - \lambda p_3 = 0 \\
\frac{\partial L}{\partial \lambda} &= w - p_1x_1 - p_2x_2 - p_3x_3 = 0
\end{aligned}$$

In the case of interior solutions, solving for λ yields the following relations

$$\begin{aligned}
\frac{x_2}{x_1} &= \frac{p_1}{p_2} \Longleftrightarrow \frac{p_2x_2}{p_1} = x_1 \\
\frac{x_3}{x_2} &= \frac{p_2}{p_3} \Longleftrightarrow x_3 = \frac{p_2x_2}{p_3} \\
\frac{x_2x_3}{x_1x_2} &= \frac{p_1}{p_3}
\end{aligned}$$

Substituting the above conditions into the budget constraint gives

$$\begin{aligned}
p_1x_1 + p_2x_2 + p_3x_3 &= \\
p_1\underbrace{\frac{p_2x_2}{p_1}}_{x_1} + p_2x_2 + p_3\underbrace{\frac{p_2x_2}{p_3}}_{x_3} &= w
\end{aligned}$$

Finally, solving for x_2 yields the Walrasian demand for good x_2,

$$x_2(w, p_1, p_2, p_3) = \frac{w}{3p_2}.$$

Similar manipulations gives the Walrasian demands for goods x_1 and x_3,

$$\begin{aligned}
x_1(w, p_1, p_2, p_3) &= \frac{w}{3p_1} \\
x_3(w, p_1, p_2, p_3) &= \frac{w}{3p_3}
\end{aligned}$$

(b) Let $x_1 + \frac{p_2}{p_1}x_2 = x_c$ denote the units of a composite good. Set up the consumer's utility maximization problem again, but now in terms of the composite good x_c. Find the Walrasian demand function for the composite good x_c.

- Since $x_1 + \frac{p_2}{p_1}x_2 = x_c$, we can express x_1 as $x_1 = x_c - \frac{p_2}{p_1}x_2$. The consumer then solves

$$\max_{x_1,x_2,x_3} \overbrace{\left(x_c - \frac{p_2}{p_1}x_2\right)}^{x_1} x_2 x_3$$
$$\text{s.t.} \quad p_1 x_c + p_3 x_3 \leq w$$

Setting up the Lagrangian, we write

$$L = (x_c - \frac{p_2}{p_1}x_2)x_2x_3 + \lambda(w - p_1x_c - p_3x_3)$$

which yields the first-order conditions

$$\begin{aligned}
\frac{\partial L}{\partial x_c} &= x_2x_3 - \lambda p_1 = 0 \\
\frac{\partial L}{\partial x_2} &= -\left(\frac{p_2}{p_1}\right)x_2x_3 + \left(x_c - \frac{p_2}{p_1}x_2\right)x_3 = 0 \\
\frac{\partial L}{\partial x_3} &= \left(x_c - \frac{p_2}{p_1}x_2\right)x_2 - \lambda p_3 = 0 \\
\frac{\partial L}{\partial \lambda} &= w - p_1x_c - p_3x_3 = 0
\end{aligned}$$

From the second first-order condition we obtain

$$x_2 = x_c\frac{p_1}{2p_2}$$

Combining first and third first-order conditions gives

$$x_3 = \frac{\left(x_c - \frac{p_2}{p_1}x_2\right)p_1}{p_3} = x_c\frac{p_1}{2p_3}$$

Substituting the expression for x_3 into the budget constraint yields the Walrasian demand for good x_c

$$x_c = \frac{2w}{3p_1}$$

which entails that the Walrasian demands for goods 2 and 3 are

$$\begin{aligned}
x_2 &= x_c\frac{p_1}{2p_2} = \frac{2w}{2p_1}\frac{p_1}{2p_2} = \frac{w}{3p_2} \\
x_3 &= x_c\frac{p_1}{2p_3} = \frac{2w}{3p_1}\frac{p_1}{2p_3} = \frac{w}{3p_3}
\end{aligned}$$

(c) Show that the Walrasian demands you found in parts (a) and (b) are equivalent.

- As shown in part (b), the Walrasian demands for good 2 and 3 coincide with those found in part (a). Regarding the Walrasian demand for good 1, we can also confirm this coincidence, as follows

$$\begin{aligned} x_1 &= x_c - \frac{p_2}{p_1} x_2 \\ &= \frac{2w}{3p_1} - \frac{p_2}{p_1} \underbrace{\frac{w}{3p_2}}_{x_2} \\ &= \frac{w}{3p_1}. \end{aligned}$$

2. Consider that the consumer exhibits Cobb-Douglas utility function $u(x_i, x_j) = x_i^\alpha x_j^\beta$, where $\alpha, \beta > 0$. For simplicity, you can assume that $\alpha + \beta = 1$.

   (a) Find his Walrasian demand for each good.

   - The consumer's Utility maximization problem

$$\max_{x_1, x_2} \quad x_i^\alpha x_j^\beta$$
$$\text{s.t. } p_i x_i + p_j x_j = w$$

   - Using the tangency condition $MRS = \frac{p_i}{p_j}$, we obtain $\frac{\alpha x_j}{\beta x_i} = \frac{p_i}{p_j}$.
     Combining the tangency condition with the budget constraint yields Walrasian demands

$$\begin{aligned} x_i(p_i, p_j, w) &= \frac{\alpha w}{(\alpha+\beta)p_i} = \frac{\alpha w}{p_i} \\ x_j(p_i, p_j, w) &= \frac{\beta w}{(\alpha+\beta)p_j} = \frac{\beta w}{p_j} \end{aligned}$$

   (b) Identify the own-price elasticity $\varepsilon_{ii} \equiv \frac{\partial x_i(p,w)}{\partial p_i} \frac{p_i}{x_i(p,w)}$, the cross-price elasticity $\varepsilon_{ij} \equiv \frac{\partial x_i(p,w)}{\partial p_j} \frac{p_j}{x_i(p,w)}$, the income elasticity $\eta_i \equiv \frac{\partial x_i(p,w)}{\partial w} \frac{w}{x_i(p,w)}$ for the Walrasian demand $x_i(p, w)$ you found in part (a) of the exercise. Interpret your results.

   - Own-price elasticity

$$\varepsilon_{ii} = \frac{\partial x_i(p,w)}{\partial p_i} \frac{p_i}{x_i(p,w)} = -\frac{\alpha w}{p_i^2} \frac{p_i^2}{\alpha w} = -1$$

     This means that when the price for good i increases by 1%, demand for good i decreases by 1%.

   - Cross-price elasticity

$$\varepsilon_{ij} = \frac{\partial x_i(p,w)}{\partial p_j} \frac{p_j}{x_i(p,w)} = 0 \times \frac{p_j p_i}{\alpha w} = 0$$

     Quantity demanded for good i does not change in response to change in price of good j.

- The income elasticity

$$\eta_i = \frac{\partial x_i(p,w)}{\partial w} \frac{w}{x_i(p,w)} = \frac{\alpha}{p_i} \frac{wp_i}{\alpha w} = 1$$

A 1% increase in consumer's income leads to an increase in quantity demanded for good i by 1%.

(c) Find his Hicksian (compensated) demand, $h_i(p,u)$, for every good $i = \{1,2\}$.

- We solve the expenditure minimization problem

$$\min_{x_1,x_2} \quad p_i x_i + p_j x_j$$
$$\text{s.t.} \quad x_i^\alpha x_j^\beta = \bar{u}$$

The Lagrangian of the problem is

$$L = p_i x_i + p_j x_j + \lambda(\bar{u} - x_i^\alpha x_j^\beta)$$

Taking first-order conditions we obtain

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= p_i - \lambda \alpha x_i^{\alpha-1} x_j^\beta = 0 \\ \frac{\partial L}{\partial x_j} &= p_j - \lambda \beta x_i^\alpha x_j \beta - 1 \\ \frac{\partial L}{\partial \lambda} &= \bar{u} - x_i^\alpha x_j^\beta = 0 \end{aligned}$$

Simultaneously solving first two first-order conditions, we find

$$\frac{\alpha x_j}{\beta x_i} = \frac{p_i}{p_j}$$

implying that

$$x_j = \frac{\beta x_i p_i}{\alpha p_j}$$

This condition coincides with the tangency condition found in part (a) of the exercise where we solved the utility maximization problem.

Substituting the expression for x_j into the constraint gives Hicksian demands defined as

$$\begin{aligned} h_i(p_i,p_j,\bar{u}) &= \left(\frac{\alpha p_j}{\beta p_i}\right)^\beta \bar{u} \\ h_j(p_i,p_j,\bar{u}) &= \left(\frac{\beta p_i}{\alpha p_j}\right)^\alpha \bar{u} \end{aligned}$$

(d) Identify the own-price elasticity, $\varepsilon_{ii}^C \equiv \frac{\partial h_i(p,w)}{\partial p_i} \frac{p_i}{h_i(p,w)}$, and cross-price elasticity, $\varepsilon_{ij}^C \equiv \frac{\partial h_i(p,w)}{\partial p_j} \frac{p_j}{h_i(p,w)}$, using the Hicksian (compensated) demand, where the superscript C denotes "compensated" demand. Interpret your findings.

- Own-price elasticity

$$\begin{aligned}
\varepsilon_{ii}^{C} &= \frac{\partial h_i(p,w)}{\partial p_i}\frac{p_i}{h_i(p,w)} \\
&= -\beta\left(\frac{\alpha p_j}{\beta p_i}\right)^{\beta-1}\bar{u}\left(\frac{\alpha p_j}{\beta p_i^2}\right)\frac{p_i}{\left(\frac{\alpha p_j}{\beta p_i}\right)^{\beta}\bar{u}} \\
&= -\beta\left(\frac{\alpha p_j}{\beta p_i^2}\right)\frac{p_i \beta p_i^2}{\alpha p_j} \\
&= \beta
\end{aligned}$$

This means that a 1% increase in own price leads to a $\left(\frac{1}{\alpha+1}\right)$ % decrease in quantity demanded for good i.

- Cross-price elasticity

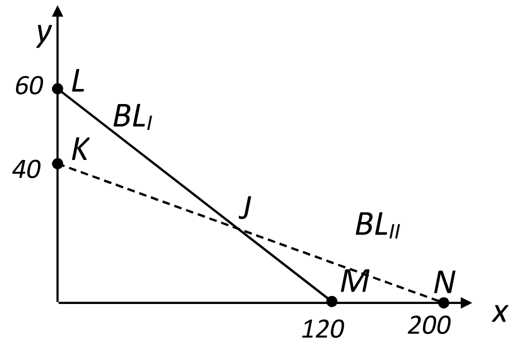
$$\begin{aligned}
\varepsilon_{ij}^{C} &= \frac{\partial h_i(p,w)}{\partial p_j}\frac{p_j}{h_i(p,w)} \\
&= \beta\left(\frac{\alpha p_j}{\beta p_i}\right)^{\beta-1}\bar{u}\left(\frac{\alpha}{\beta p_i}\right)\frac{p_j}{\left(\frac{\alpha p_j}{\beta p_i}\right)^{\beta}\bar{u}} \\
&= \beta\left(\frac{\alpha \bar{u}}{\beta p_i}\right)\frac{p_j \beta p_i}{\alpha \bar{u} p_j} \\
&= \beta
\end{aligned}$$

The cross-price elasticity is positive which means that goods are substitutes. The quantity demanded for good j increases by (β) % when the price of good i rises by 1%..

3. Fran has a monthly income of \$60. She spends her money making telephone calls (measured in minutes) at a price p_x and on other composite good y, whose price has been normalized to one, i.e., $p_y = \$1$. Her mobile phone company offers her two plans: Plan I, in which she pays no monthly fee and makes calls for \$0.50 per minute; or Plan II in which she pays a \$20 monthly fee and benefits from cheaper phone calls at \$0.20 per minute.

   (a) Depict Fran's budget constraint under each of the two plans, with the number of phone calls (good x) in the horizontal axis and the composite good (good y) in the vertical. Identify the intersection point between these two plans.

(i) Let x denote the number of phone calls, and y denote spending on other goods. The expression of the budge line under Plan I, BL_I is $0.5x + y = 60$, or $y = 60 - 0.5x$, as depicted in the solid line of the following figure that originates at $y = 60$ and which crosses the horizontal axis at $x = 120$. Under Plan II, Fran's budget line, BL_{II}, is $0.2x + y = 40$, or $y = 40 - 0.2x$, as illustrated in the figure by the dashed line that originates at $y = 40$ and crosses the horizontal axis at $x = 200$. These two budge lines intersect each other at $0.5x + (40 - 0.2x) = 60$, i.e., $x = 66.67$. Hence, y$= 40 - 0.2x = 40 - (0.2 \times 66.67) = 26.67$. Therefore, BL_I and BL_{II} intersect at bundle $(66.67, 26.67)$.



1. b. If Fran decides that plan I is better for her, what is the set of baskets she may purchase if her behavior is consistent with WARP.

According to WARP, if the consumption bundle under new prices and wealth was affordable under the original prices and wealth, $px(p', w',) \leq w$, then the bundle the decision makes selected under the old prices and wealth cannot be affordable under the new prices and wealth, i.e., $p'x(p,, w,) \leq w'$. In this context, where the consumer moves from facing budget line BL_{II} to BL_I, WARP states that, if the consumption bundle under BL_I, $x(p', w')$, is affordable under BL_I, it must lie on segment KJ in the above figure, i.e., this is equivalent to the premise of WARP, $px(p', w',) \leq w$. Hence, the bundle selected when facing budget line BL_I, $x(p, w)$, must be unaffordable under BLB; that is, $x(p, w)$ must lie on segment LJ of budget line BL_I. Notice that bundles in segment JM are instead affordable under BL_{II}, thus violating WARP.

4. Consider a continuous and strictly increasing utility function $u : \mathbb{R}^N_+ \to \mathbb{R}$, and a vector of positive prices $p >> 0$.

   (a) Show that the expenditure function $e(p, u)$ is concave in prices.

   - Using a similar notation as in part (b) of Exercise 22, concavity of the expenditure function can be formally expressed as

$$\alpha e(p^1, u) + (1 - \alpha)e(p^2, u) \leq e(p^\alpha, u)$$

where $p^\alpha \equiv \alpha p^1 + (1 - \alpha)p^2$ and $\alpha \in (0, 1)$. Intuitively, this property entails that the consumer's minimal expenditure is lower when facing a extreme budget set than with the average of the two. By expenditure minimization, bundle x^1 solves EMP when facing price vector p^1, i.e., $p^1 \cdot x^1 \leq p^1 \cdot x$ for all $x \in B^1$. A similar argument applies when the price vector changes to p^2, i.e, the expenditure-minimizing bundle x^2 satisfies $p^2 \cdot x^2 \leq p^2 \cdot x$ for all $x \in B^2$. Since these two inequalities hold for any feasible x, they must also hold true for a specific bundle $\overline{x}$, that is $p^1 \cdot x^1 \leq p^1 \cdot \overline{x}$ and $p^2 \cdot x^2 \leq p^2 \cdot \overline{x}$. Multiplying the first inequality by α and the second by $(1 - \alpha)$, we obtain

$$\begin{aligned} \alpha p^1 \cdot x^1 &\leq \alpha p^1 \cdot \overline{x} \quad \text{and} \\ (1-\alpha)\, p^2 \cdot x^2 &\leq (1-\alpha)\, p^2 \cdot \overline{x} \end{aligned}$$

and adding them up yields

$$\alpha p^1 \cdot x^1 + (1-\alpha)\, p^2 \cdot x^2 \leq \alpha p^1 \cdot \overline{x} + (1-\alpha)\, p^2 \cdot \overline{x}$$

which, rearranging, and noting that $p^1 \cdot x^1 \equiv e(p^1, u)$ and $p^2 \cdot x^2 \equiv e(p^2, u)$

$$\alpha \underbrace{p^1 \cdot x^1}_{e(p^1,u)} + (1-\alpha) \underbrace{p^2 \cdot x^2}_{e(p^2,u)} \;\leq\; \underbrace{\left[\alpha p^1 + (1-\alpha)\, p^2\right]}_{p^\alpha} \cdot \overline{x}$$

that is,

$$\alpha e(p^1, u) + (1-\alpha) e(p^2, u) \leq e(p^\alpha, u)$$

as required. Hence, the expenditure function is concave in prices. (For a graphical representation of this property, see figure 2.41 in the textbook.)

(b) Show that the expenditure function $e(p, u)$ is strictly increasing in the utility level that the consumer seeks to reach, u.

- Before presenting a formal proof, here is some intuition: When solving the EMP, the consumer chooses a bundle x satisfying the constraint $u(x) \geq u$ with equality. (Otherwise, he could choose a cheaper bundle that still satisfies $u(x) \geq u$.) If the target utility level u increases, then the consumer needs to purchase more units of at least one good, thus raising his minimal expenditure $e(p, u)$. We next present a detailed proof of this property.
- *Plan of the proof.* Let us work by contradiction, assuming that $e(p, u)$ is *not* strictly increasing in u. Hence, as a premise, consider two different utility levels $u^B > u^A$, and let x^A and x^B be the optimal consumption bundles (those solving the EMP) when the utility level that the consumer seeks to reach is u^A and u^B, respectively, that is,

$$x^A \in h(p, u^A) \quad \text{and} \quad x^B \in h(p, u^B)$$

By contradiction, let us assume that the minimal expenditure $e(p, u)$ was *not* increasing in u, thus implying that $p \cdot x^A \geq p \cdot x^B$. In words, the minimal expenditure that the consumer needs to incur when purchasing the optimal bundle that reaches the high utility level u^B is lower or equal than that when he purchases the optimal bundle that reaches the lower utility level u^A.

- *Proof.* Let us consider a "scaled-down" version of bundle x^B, where all components are reduced by a common factor λ, i.e., $\widehat{x}^B \equiv \lambda x^B$ where $\lambda \in (0, 1)$. Note that such a bundle is strictly cheaper than bundle x^B, since it contains fewer units of all goods than bundle x^B, thus implying that

$$p \cdot x^A \geq p \cdot x^B > p \cdot \widehat{x}^B$$

Since the utility function $u(x)$ is continuous, we can find make bundle $\widehat{x}^B$ close enough to x^B (i.e., approaching λ to 1) such that its associated utility level $u(\widehat{x}^B)$ is close to that of bundle x^B, i.e., $u(\widehat{x}^B)$ is close to $u(x^B) = u^B$. In addition, since $u^B > u^A$ by assumption, then $u(\widehat{x}^B) > u^A$. However, as described above, the minimal expenditures in this context satisfies $p \cdot x^A >$

$p \cdot \widehat{x}^B$, which contradicts that bundle x^A could solve the EMP when reaching for utility level u^A. (The consumer could have reached utility level u^A by purchasing the cheaper bundle $\widehat{x}^B$ rather than x^A.) We have thus reached a contradiction, implying that expenditure function $e(p, u)$ must be strictly increasing in u.

- Intuitively, in order to reach a higher utility level the consumer needs to purchase larger amounts of at least one of the goods, and thus his minimal expenditure raises in u. (For a graphical representation, see figure 2.40 in the textbook.)