Homework #5 (Due on October 1st, 2018)

1. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce one type of output. The price of every unit of input is $w = 8$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem, and solve for its unconditional factor demand $z(8, p)$.

   (b) Evaluate the profit function at the unconditional factor demand $z(8, p)$. Test for convexity of the profit function in output price p.

   (c) Let us now illustrate convexity in output prices by using an alternative approach: (1) evaluate the profit function you found in part (b) at prices $p = 6$, and at $p = 12$. Then, find their convex combination $\alpha\pi(6) + (1 - \alpha)\,\pi(12)$ where $\alpha \in [0, 1]$; (2) evaluate the profit function at the convex combination of the above output prices, that is, $\pi(\alpha 6 + (1 - \alpha)\,12)$. Last, show that the profit function you found in step (1) lies weakly above that found in step (2) for all values of α, that is,

$$\alpha\pi(6) + (1 - \alpha)\,\pi(12) \geq \pi(\alpha 6 + (1 - \alpha)\,12).$$

2. Consider a firm with production function $q = \sqrt{z}$, using one input (e.g., labor) to produce units of output q. The price of every unit of input is $w > 0$, and the price of every unit of output is $p > 0$.

   (a) Set up the firm's profit-maximization problem (PMP), and solve for its unconditional factor demand $z(w, p)$.

   (b) What is the output level that arises from using the amount of inputs $z(w, p)$? Label this output level $q(w)$.

   (c) Set up the firm's cost-minimization problem (CMP), and solve for its conditional factor demand $z(w, q)$ for any output level q. (For now, we write the constraint of the CMP to be $f(z) \geq q$, where the output level q that the firm seeks to reach does not necessarily coincide with that found in part (b), $q(w)$.)

   (d) Evaluate the conditional factor demand $z(w, q)$ at output level $q = q(w)$, to obtain $z(w, q(w))$. Show that it coincides with the unconditional factor demand $z(w, p)$ found in part (a), that is,

$$z(w, q(w)) = z(w, p).$$

(e) *Shephard's lemma.* Evaluate the CMP's objective function, $w \cdot z$, at the conditional factor demand $z(w, q)$, to obtain the cost function, that is, find $c(w, q) = w \cdot z(w, q)$. Differentiate the cost function with respect to w, and show that your result coincides with the conditional factor demand $z(w, q)$.

(f) *Substitution and output effects.* Let us now consider that the firm faces cheaper wages (lower w). Differentiate the unconditional factor demand $z(w, p)$ found in part (a) with respect to w to find the total effect of this price change.

(g) Differentiate the conditional factor demand $z(w, q)$ found in part (c) with respect to w to obtain the substitution effect of this price change.

(h) Compare your results in parts (f) and (g). Which is the output effect of the change in w?