Solution - Recitation 1, Fall 2018

1. **Checking properties of preference relations-I.** For each of the following preference relations in the consumption of two goods (1 and 2): describe the upper contour set, the lower contour set, the indifference set of bundle (2,1), and interpret them. Then check whether these preference relations are rational (by separately examining whether they are complete and transitive), monotone, and convex.

   (a) Bundle (x_1, x_2) is weakly preferred to (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$ if and only if $x_1 \geqslant y_1 - 1$.

   - Let us first build some intuition on this preference relation. First, note that an individual prefers a bundle x to another bundle y if and only if the first component of bundle x, x_1, contains at least one unit less than the first component of bundle y, i.e., $x_1 \geqslant y_1 - 1$. For instance, (2, 1) is preferred to (2, 6) since $x_1 = 2$ and $y_1 = 2$, thus implying $2 \geqslant 2 - 1 = 1$. Importantly, the individual ignores the content of the second component when comparing two bundles. Let us next describe the upper contour, lower contour, and indifference set of a given bundle, such as (2, 1). You can take any other bundle of course! The upper contour set of this bundle is given by

$$UCS(2,1) = \{(x_1, x_2) \succsim (2,1) \Longleftrightarrow x_1 \geq 2 - 1\} = \{(x_1, x_2) : x_1 \geq 1\}$$

     while the lower contour set is defined as

$$LCS(2,1) = \{(2,1) \succsim (x_1, x_2) \Longleftrightarrow 2 \geq x_1 - 1\} = \{(x_1, x_2) : x_1 \leq 3\}$$

     Finally, the consumer is indifferent between bundle (2,1) and the set of bundles where

$$IND(2,1) = \{(x_1, x_2) \sim (2,1) \Longleftrightarrow 1 \leq x_1 \leq 3\}$$

     Figure 1.1 depicts:

     1. All bundles in $\mathbb{R}^2_+$ such that $x_1 \geq 1$, and thus belong to the $UCS(2,1)$, i.e., the set of bundles that are weakly preferred to (2,1);
     2. All bundles such that $x_1 \leq 3$ and are therefore in the $LCS(2,1)$, i.e., the set of bundles weakly preferred by (2,1); and

3. Those bundles in between, $1 \leq x_1 \leq 3$, in the $IND(2,1)$ are indifferent between them and bundle (2,1). [Figure 1.1 represents the UCS, LCS and IND sets, where all cutoffs we found were on the x_1-axis, since this individual ignores the amount of x_2.]

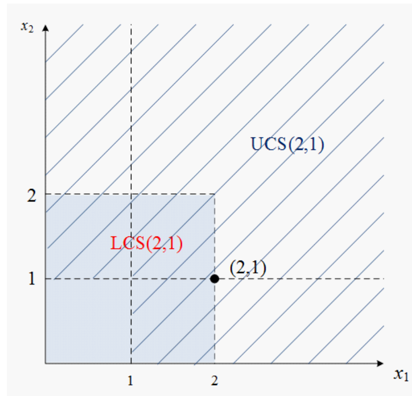

Figure 1.1. UCS, LCS, and IND of bundle (2,1).

As a remark, this preference relation satisfies *continuity*. In particular, continuity requires that both the upper and the lower contour sets are closed, which is satisfied given that they both contain their boundary points.

- *Completeness*. For this property to hold, we need that, for any pair of bundles (x_1, x_2) and (y_1, y_2), either $(x_1, x_2) \succsim (y_1, y_2)$ or $(y_1, y_2) \succsim (x_1, x_2)$, or both (i.e., $(x_1, x_2) \sim (y_1, y_2)$). Since this preference relation only depends on the first component of every bundle, we have that, for every pair of bundles (x_1, x_2) and (y_1, y_2), either:

  1. $x_1 \geq y_1 - 1$, which implies that $(x_1, x_2) \succsim (y_1, y_2)$; or
  2. $x_1 < y_1 - 1$, which implies

$$y_1 > x_1 + 1 > x_1 - 1,$$

  and hence $y_1 > x_1 - 1$, thus ultimately yielding $(y_1, y_2) \succsim (x_1, x_2)$. Hence, this preference relation is complete.

  Additionally, note that this preference relation satisfies reflexivity, since completeness implies reflexivity, i.e., every bundle (x_1, x_2) is weakly preferred to itself.

- *Transitivity*. We need to show that, for any three bundles (x_1, x_2), (y_1, y_2)

and (z_1, z_2) such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

This property does not hold for this preference relation. In order to show this result, notice that a bundle (x_1, x_2) is preferred to another bundle (y_1, y_2) if its first component, x_1, is larger than that of the other bundle, y_1, by less than one unit, i.e., condition $x_1 \geq y_1 - 1$ is equivalent to $1 \geq y_1 - x_1$. This condition about the distance between x_1 and y_1 is depicted in the bottom left-hand side of figure 1.2. A similar argument can be extended to the comparison between two bundles (y_1, y_2) and (z_1, z_2), where the former is preferred to the latter if and only if the distance between their first components is smaller than one, i.e., $1 \geq z_1 - y_1$; also depicted at the bottom of figure 1.2, but on the right-hand side. Hence, for bundle (x_1, x_2) to be preferred to (z_1, z_2), i.e., $(x_1, x_2) \succsim (z_1, z_2)$, we need that the distance between their first components is smaller than one, i.e., $1 \geq z_1 - x_1$; as we next show with a counterexample. Consider the following three bundles (notice that the second component of every bundle is inconsequential, since the preference ordering only relies on a comparison of the first component of every vector):

$$\begin{aligned}(x_1, x_2) &= (5, 4)\\ (y_1, y_2) &= (6, 1)\\ (z_1, z_2) &= (7, 2)\end{aligned}$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ since the difference in their first component is smaller (or equal) to one unit, $x_1 \geq y_1 - 1$ (i.e., $5 \geq 6 - 1$). Additionally, $(y_1, y_2) \succsim (z_1, z_2)$ is also satisfied since $y_1 \geq z_1 - 1$ (i.e., $6 \geq 7 - 1$). However, $(x_1, x_2) \not\succsim (z_1, z_2)$ since the difference between z_1 and x_1 is larger than one unit, $x_1 \ngeq z_1 - 1$ (i.e., $5 \ngeq 7 - 1$). Hence, this preference relation does not satisfy transitivity.

- *Monotonicity.* This property is satisfied for this preference relation. In particular, increasing the amount of good 1 yields a new bundle $(x_1 + \varepsilon, x_2)$ that is weakly preferred to the original bundle (x_1, x_2), i.e., the comparison of their first component yields $x_1 + \varepsilon \geq x_1 - 1$, or $\varepsilon \geq -1$, which holds since $\varepsilon > 0$ by assumption. Similarly, increasing the amount of the second component produces a new bundle $(x_1, x_2 + \varepsilon)$ which is weakly preferred to the original bundle (x_1, x_2). Recall that this individual compares bundles by evaluating the first component alone. Since in this case the amount of the first component is unaffected, then he is indifferent between bundle (x_1, x_2) and $(x_1, x_2+\varepsilon)$; an indifference that is allowed by the definition of monotonicity. Hence, the preference relation satisfies monotonicity. As a curiosity, note that, while the preference relation satisfies monotonicity, it does not satisfy strong monotonicity. Indeed, for this property to hold, we need that an increase in the amount of *any* of the goods yields a new bundle that is strictly preferred to the original bundle (x_1, x_2). While this is true if we increase the amount of good 1, it is not if we only increase the amount of good 2, thus not satisfying strict monotonicity.

- *Convexity.* This property implies that the upper contour set must be convex, that is, if bundle (x_1, x_2) is weakly preferred to (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$, then the convex combination of these two bundles is also weakly preferred to (y_1, y_2),

$$\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2) \quad \text{for any } \lambda \in [0, 1]$$

In this case, $(x_1, x_2) \succsim (y_1, y_2)$ implies that $x_1 \geq y_1 - 1$; whereas $\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2)$ implies

$$\lambda x_1 + (1 - \lambda) y_1 \geq y_1 - 1$$

which simplifies to $\lambda x_1 \geq \lambda y_1 - 1$. However, the premise from $(x_1, x_2) \succsim (y_1, y_2)$, i.e., $x_1 \geq y_1 - 1$, entails that $\lambda x_1 \geq \lambda y_1 - 1$ must also hold. (To see that, note that $x_1 \geq y_1 - 1$ can be written as $(x_1 - y_1) + 1 \geq 0$, while $\lambda x_1 \geq \lambda y_1 - 1$ can be expressed as $\lambda (x_1 - y_1) + 1 \geq 0$, where $\lambda (x_1 - y_1) + 1 \leq (x_1 - y_1) + 1$ since $\lambda \in [0, 1]$.) Therefore, $(x_1 - y_1) + 1 \geq 0$ is a sufficient condition for $\lambda (x_1 - y_1) + 1 \geq 0$, ultimately implying that $\lambda x_1 \geq \lambda y_1 - 1$ must hold. Hence, this preference relation is convex.

(b) Bundle (x_1, x_2) is weakly preferred to (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if $x_1 \geq y_1 - 1$ and $x_2 \leq y_2 + 1$.

- Let us first build some intuition on this preference relation. Similarly as the preference relation we first analyzed, the individual prefers bundle x to y if the first component of x is larger than that of y in less than one unit, i.e., $x_1 \geqslant y_1 + 1$ or $1 \geqslant y_1 - x_1$; but in addition, he must find that the difference between their second components is larger than one unit, i.e., $x_2 \leqslant y_2 + 1$ or $1 \leqslant y_2 - x_2$.

- Let us next find the upper contour, lower contour, and indifference set of a given bundle, such as (2, 1). The upper contour set of this bundle is given by

$$\begin{aligned} UCS(2,1) &= \{(x_1, x_2) \succsim (2,1) \Longleftrightarrow x_1 \geq 2-1 \text{ and } x_2 \leq 1+1\} \\ &= \{(x_1, x_2) : x_1 \geq 1 \text{ and } x_2 \leq 2\} \end{aligned}$$

which is graphically represented in figure 1.2 by all those bundles in the lower right-hand corner (below $x_2 = 2$ and to the right of $x_1 = 1$). On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(2,1) &= \{(2,1) \succsim (x_1, x_2) \Longleftrightarrow 2 \geq x_1 - 1 \text{ and } 1 \leq x_2 + 1\} \\ &= \{(x_1, x_2) : x_1 \leq 3 \text{ and } x_2 \geq 0\} \end{aligned}$$

which is depicted in figure 1.2 by all those bundles in the left half of the positive quadrant (above $x_2 = 0$ and to the left of $x_1 = 3$). Finally, the consumer is indifferent between bundle (2,1) and the set of bundles where

$$IND(2,1) = \{(x_1, x_2) \sim (2,1) \Longleftrightarrow 1 \leq x_1 \leq 3 \text{ and } 0 \leq x_2 \leq 2\}$$

graphically, represented by the rectangle in the bottom center of the figure, for all $x_2 \leq 2$ and $1 \leq x_1 \leq 3$.

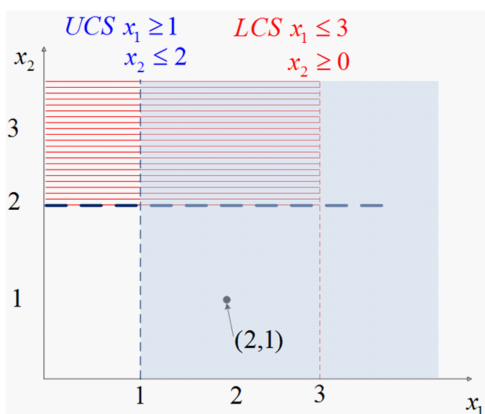

Figure 1.2. UCS, LCS and IND of bundle (2,1).

- *Completeness.* From the above analysis it is easy to note that completeness is *not* satisfied, since there are bundles in the area $x_1 > 3$ and $x_2 > 2$ where our preference relation does not specify if they belong to the upper contour set, the lower contour set, or the indifference set of bundle (2, 1). Hence, any bundle in the unshaded region where $x_1 > 3$ and $x_2 > 2$ (in the top right-hand corner of figure 1.2) would be incomparable with (2, 1). Another way to prove that completeness does not hold is by finding a counterexample. In particular, we must find an example of two bundles such that neither $(x_1, x_2) \succsim (y_1, y_2)$ nor $(y_1, y_2) \succsim (x_1, x_2)$. Let us take, for instance, two bundles,

$$(x_1, x_2) = (1, 2) \text{ and } (y_1, y_2) = (4, 6)$$

We have that:

1. $(x_1, x_2) \not\succsim (y_1, y_2)$ since $1 \ngeq 4 - 1$ for the first component of the bundle (and we need $x_1 \geq y_n - 1$ for $(x_1, x_2) \succsim (y_1, y_2)$ to hold), and
2. $(y_1, y_2) \not\succsim (x_1, x_2)$ since $6 \nleq 2+1$ for the second component of the bundle. Hence, for these are two bundles neither $(x_1, x_2) \succsim (y_1, y_2)$ nor $(y_1, y_2) \succsim (x_1, x_2)$, which implies that this preference relation is not complete.

- *Transitivity.* We need to show that, for any three bundles (x_1, x_2), (y_1, y_2) and (z_1, z_2) such that

$$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

This property does not hold for this preference relation. In order to show that, let us consider the following three bundles (that is, we are finding a counterexample to show that transitivity does not hold):

$$\begin{aligned} (x_1, x_2) &= (2, 1) \\ (y_1, y_2) &= (3, 4) \\ (z_1, z_2) &= (4, 6) \end{aligned}$$

First, note that $(x_1, x_2) \succsim (y_1, y_2)$ since the distance between their first components is not larger than one unit $x_1 \geq y_1 - 1$ (i.e., $2 \geq 3 - 1$), and the distance between the second components is larger than one unit $x_2 \leq y_2 + 1$ (i.e., $1 \leq 4 + 1$). Additionally, $(y_1, y_2) \succsim (z_1, z_2)$ is also satisfied since $y_1 \geq z_1 - 1$ for the first component (i.e., $3 \geq 4 - 1$), and $y_2 \leq z_2 + 1$ for the second component (i.e., $3 \leq 4+1$). However, $(x_1, x_2) \not\succsim (z_1, z_2)$ since the difference of the first components is strictly larger than one unit $x_1 \ngeq z_1 - 1$ (i.e., $2 \ngeq 4 - 1$). Hence, this preference relation does not satisfy Transitivity.

- *Monotonicity.* For this property to hold, we need that an increase in the amounts of one good yields a new bundle that is weakly preferred to the original bundle. Indeed, if we increase the amount of good 1 by $\varepsilon > 0$ to create bundle $(x_1+\varepsilon, x_2)$, we have that the first component satisfies $x_1+\varepsilon \geqslant x_1-1$, i.e., $\varepsilon \geqslant -1$, and the second component satisfies $x_2 \leqslant x_2+1$, i.e., $0 \leqslant 1$. If we only increase the amounts of good 2, a similar argument applies. Finally, if we increase the amounts of both goods 1 and 2 simultaneously, according to the definition of monotonicity we need that the newly created bundle is strictly preferred to the initial bundle, i.e., $(x_1+\varepsilon, x_2+\delta) \succ (x_1, x_2)$ where constants $\varepsilon, \delta > 0$ are allowed to differ for each good. For this relationship to hold, note that we need that: (1) the first components satisfy $x_1+\varepsilon \geq x_1-1$, or $\varepsilon \geq -1$ (which holds by definition); and (2) the second components satisfy $x_2+\delta \leq x_2+1$, which implies $\delta \leq 1$ (which does not necessarily hold by assumption). Therefore, for this preference relation to be monotonic, we need that $\delta \leq 1$. In other words, if good 2 is increased by more than one unit, the preference relation is not monotonic.

For instance, if the amount of both goods is increased by two units, i.e., $\varepsilon = \delta = 2$, the new bundle (x_1+2, x_2+2) is not necessarily preferred to the original bundle (x_1, x_2) since the condition on the first component, $x_1+2 \geq x_1-1$, holds but that on the second component, $x_2+2 \leq x_2+1$, does not.

- *Convexity.* This property implies that the upper contour set must be convex. That is, if bundle (x_1, x_2) is weakly preferred to (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$, then the convex combination of these two bundles is also weakly preferred to (y_1, y_2),

$$\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) \succsim (y_1, y_2) \quad \text{for any } \lambda \in [0,1]$$

In this case, $(x_1, x_2) \succsim (y_1, y_2)$ implies that $x_1 \geq y_1 - 1$ and $x_2 \leq y_2 + 1$; whereas $\lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) \succsim (y_1, y_2)$ implies

$$\begin{aligned} \lambda x_1 + (1-\lambda) y_1 &\geq y_1 - 1 \quad \text{for the first component, and} \\ \lambda x_2 + (1-\lambda) y_2 &\leq y_2 + 1 \quad \text{for the second component;} \end{aligned}$$

which respectively can be rewritten as

$$\begin{aligned} \lambda(x_1 - y_1) &\geq -1, \text{ and} \\ \lambda(x_2 - y_2) &\leq 1 \end{aligned}$$

In addition, the condition on the first component $x_1 - y_1 \geqslant -1$ (or alternatively, $x_1 \geqslant y_1 - 1$) holds by assumption since $(x_1, x_2) \succsim (y_1, y_2)$. Similarly, the condition on the second component $x_2 - y_2 \leqslant 1$ (or alternatively, $x_2 \leqslant y_2 + 1$) is also satisfied by $(x_1, x_2) \succsim (y_1, y_2)$. Hence, the preference relation satisfies convexity.

2. **Lexicographic preference relation.** Let us define a lexicographic preference relation in a consumption set $X \times Y$, as follows:

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if and only if } \begin{cases} x_1 > y_1, \text{ or if} \\ x_1 = y_1 \text{ and } x_2 \geqslant y_2 \end{cases} \tag{1}$$

Intuitively, the consumer prefers bundle x to y if the former contains more units of the first good than the latter, i.e., $x_1 > y_1$. However, if both bundles contain the same amounts of good 1, $x_1 = y_1$, the consumer ranks bundle x above y if the former has more units of good 2 than the latter, i.e., $x_2 \geq y_2$. For simplicity, assume that both components have been normalized to $X = [0, 1]$ and $Y = [0, 1]$.

(a) Show that the lexicographic preference relation satisfies rationality (i.e., it is complete and transitive).

1. *Completeness.* By definition, $\succsim$ is a complete preference relation if for all bundles (x_1, x_2),$(y_1, y_2) \in \mathbb{R}^2$, either $(x_1, x_2) \succsim (y_1, y_2)$, or $(y_1, y_2) \succsim (x_1, x_2)$, or both. Hence, we need to show that

$$(x_1, x_2) \not\succsim (y_1, y_2) \implies (y_1, y_2) \succsim (x_1, x_2)$$

Indeed, note that $(x_1, x_2) \not\succsim (y_1, y_2)$ can be expressed as

$$(x_1, x_2) \not\succsim (y_1, y_2) \text{ if } \begin{cases} y_1 \geq x_1, \text{ and if} \\ y_1 \neq x_1 \text{ or } y_2 > x_2 \end{cases} \tag{2}$$

Expression (2) describes that bundle (y_1, y_2) contains weakly more units of good 1 than (x_1, x_2) does, thus implying that a consumer with a lexicographic preference relation weakly prefers (y_1, y_2) to (x_1, x_2), i.e., $(y_1, y_2) \succsim (x_1, x_2)$. Therefore, we have shown that $(x_1, x_2) \not\succsim (y_1, y_2)$ implies $(y_1, y_2) \succsim (x_1, x_2)$. Hence, the preference relation is complete.

2. *Transitivity.* Let us take three bundles (x_1, x_2),(y_1, y_2) and $(z_1, z_2) \in \mathbb{R}^2$ with $(x_1, x_2) \succsim (y_1, y_2)$:

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if and only if } \begin{cases} x_1 > y_1, \text{ or if} \\ x_1 = y_1 \text{ and } x_2 \geqslant y_2 \end{cases}$$

and $(y_1, y_2) \succsim (z_1, z_2)$, that is

$$(y_1, y_2) \succsim (z_1, z_2) \text{ if and only if } \begin{cases} y_1 > z_1, \text{ or if} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases}$$

Hence, we need to check for transitivity in the four possible cases in which $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$.

a) If $x_1 > y_1$, and $y_1 > z_1$, then by the transitivity of the "greater than or equal" operator ($\geq$), we obtain $x_1 > z_1$. As we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$, then transitivity holds in this case.

b) If ($x_1 = y_1$ and $x_2 \geqslant y_2$) and ($y_1 = z_1$ and $y_2 \geqslant z_2$), then ($x_1 = z_1$ and $x_2 \geqslant z_2$). In addition, we know that ($x_1 = z_1$ and $x_2 \geqslant z_2$) implies $(x_1, x_2) \succsim (z_1, z_2)$, which validates transitivity.

c) If $x_1 > y_1$, and ($y_1 = z_1$ and $y_2 \geqslant z_2$), then $x_1 > z_1$. As we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$. Transitivity holds in this case as well.

d) If $y_1 > z_1$ and ($x_1 = y_1$ and $x_2 \geqslant y_2$), then $x_1 > z_1$, and we know that $x_1 > z_1$ implies $(x_1, x_2) \succsim (z_1, z_2)$, entailing that transitivity holds in this case as well. We have then checked all four cases under which $(x_1, x_2) \succsim (y_1, y_2)$ and $(y_1, y_2) \succsim (z_1, z_2)$ may arise, and in all of them we obtained $(x_1, x_2) \succsim (z_1, z_2)$, confirming that this preference relation is transitive. Therefore, since the preference relation is complete and transitive, we can conclude that it is rational.

(b) Show that the lexicographic preference relation $\succsim$ *cannot* be represented by a utility function $u : X \times Y \to \mathbb{R}$.

- Let us work by contradiction. So, let us suppose that there is a utility function $u(\cdot)$ representing this lexicographic preference relation $\succsim$. Then, for any $x_1 \in X$, the pair $(x_1, 1)$ is strictly preferred to the pair $(x_1, 0)$, i.e., $(x_1, 1) \succ (x_1, 0)$. If there is a utility function $u(\cdot)$ representing this preference relation, then we must have that

$$(x_1, 1) \succ (x_1, 0) \Longleftrightarrow u(x_1, 1) > u(x_1, 0)$$

On the other hand, from the Archimedean property, we know that we can pick a rational number $r(x_1)$ such that it lies in between $u(x_1, 1)$ and $u(x_1, 0)$.

$$u(x_1, 1) > r(x_1) > u(x_1, 0)$$

Let us take any $x_1, x_2 \in X$, and let us suppose without loss of generality that

$x_1 > x_2$. Similarly to our above result, we then have that

$$u(x_2, 1) > r(x_2) > u(x_2, 0)$$

And since $x_1 > x_2$, we have that

$$u(x_1, 1) > r(x_1) > u(x_1, 0) > u(x_2, 1) > r(x_2) > u(x_2, 0)$$

which implies

$$r(x_1) > r(x_2)$$

Then, $r(\cdot)$ provides a one-to-one function from the set of real numbers, $\mathbb{R}$ (which is uncountable) to the set of rational numbers, $\mathbb{Q}$, which is countable. But this is a mathematical impossibility.[1] Thus, we conclude that there can be no utility function representing the lexicographic preferences when they are defined over a continuous set $X \times Y$, where $X = [0, 1]$ and $Y = [0, 1]$.

(c) Assume now that this preference relation is defined on a *finite* consumption set $X = X_1 \times X_2$, where $X_1 = \{x_{11}, x_{12}, ..., x_{1n}\}$ and $X_2 = \{x_{21}, x_{22}, ..., x_{2m}\}$. [*Hint*: You can define a function $N_i(x_{ij})$ as the number of elements in sequence X_i prior to element x_{ij}; that is,

$$N_i\left(x_{ij}\right) = \#\left\{y \in X_i | y < x_{ij}\right\}.$$

Then define a utility function

$$u(y_1, y_2) = mN_1(y_1) + N_2(y_2), \text{ where } m > 0,$$

and for any pair $(y_1, y_2) \in X_1 \times X_2$.]

1. Let us first define a function $N_i(x_{ij})$ as the number of elements in sequence X_i prior to element x_{ij}:

$$N_i\left(x_i\right) = \#\left\{y \in X_i | y < x_{ij}\right\}, \text{ where } X_i = \left\{x_{i1}, x_{i2}, ..., x_{in}\right\}.$$

Then, we define a utility function $u(y_1, y_2) = mN_1(y_1) + N_2(y_2)$ for any pair $(y_1, y_2) \in X_1 \times X_2$. In order to show that this utility function indeed represents the lexicographic preference relation (when consumption sets are finite), we

[1] For a review of real and rational numbers, see, for instance, Simon and Blume's *Mathematics for Economists*, pp. 848-849

need to show the usual two lines of implication:

$$(y_1, y_2) \succsim (z_1, z_2) \Longrightarrow u(y_1, y_2) \geq u(z_1, z_2), \text{ and}$$

$$(y_1, y_2) \succsim (z_1, z_2) \Longleftarrow u(y_1, y_2) \geq u(z_1, z_2)$$

2. Let us first show that $(y_1, y_2) \succsim (z_1, z_2) \Longrightarrow u(y_1, y_2) \geq u(z_1, z_2)$. In order to show this result, we need that

$$\begin{cases} y_1 > z_1, \text{ or} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases} \quad \text{implies } mN_1(y_1) + N_2(y_2) \geq mN_1(z_1) + N_2(z_2)$$

Hence, we first need to check if this inequality is satisfied when $y_1 > z_1$, and when ($y_1 = z_1$ and $y_2 \geqslant z_2$).

a) Let us first check if $y_1 > z_1$ implies $mN_1(y_1)+N_2(y_2) \geq mN_1(z_1)+N_2(z_2)$. Alternatively, we can rewrite this inequality as

$$m\underbrace{[N_1(y_1) - N_1(z_1)]}_{a} + \underbrace{[N_2(y_2) - N_2(z_2)]}_{b} \geq 0 \tag{1}$$

Let us analyze if this expression can ever be negative (we will examine the infimum values) by separately evaluating the infimum of terms (a) and (b). Regarding term (a), we know that, if $y_1 > z_1$,

$$\inf [N_1(y_1) - N_1(z_1)] = k - (k-1) = 1,$$

$$\text{since } N_1(y_1) > N_1(z_1) \text{ given that } y_1 > z_1$$

and hence, $\inf [m [N_1(y_1) - N_1(z_1)]] = m$. Thus, $m [N_1(y_1) - N_1(z_1)] \geq m$, and term (a) in expression (1) is always weakly above m. Let us now focus on term (b) of expression (1):[2]

$$\inf [N_2(y_2) - N_2(z_2)] = \inf N_2(y_2) - \sup N_2(z_2) = 0 - (m-1) = 1 - m$$

Intuitively, the result $\inf N_2(y_2) = 0$ implies that there are no elements prior to y_2 (that is, y_2 is the first term of the sequence); in contrast, $\sup N_2(z_2) = m - 1$ means that z_2 is the last element in the sequence of length m, and hence all other $m - 1$ elements in the sequence were located prior to z_2. Hence, $N_1(y_1) - N_1(z_1) \geq 1 - m$, and thus term (b)

[2]Note that we are not imposing any conditions on y_2 and z_2, since we only assumed that $y_1 > z_1$.

in expression (1) always lies above $1 - m$. Combining the results of the first and second term of the infimum of expression (1), we can conclude that

$$m\left[N_1(y_1) - N_1(z_1)\right] + \left[N_2(y_2) - N_2(z_2)\right] \geq m - (1 - m) = 1$$

which is clearly above 0. Recall that we needed to show that

$$m\left[N_1(y_1) - N_1(z_1)\right] + \left[N_2(y_2) - N_2(z_2)\right] \geq 0.$$

Therefore, $y_1 > z_1$ indeed implies $u(y_1, y_2) \geq u(z_1, z_2)$.

b) Let us now check that ($y_1 = z_1$ and $y_2 \geqslant z_2$) also implies $mN_1(y_1) + N_2(y_2) \geq mN_1(z_1) + N_2(z_2)$. Alternatively, we can rewrite this inequality as

$$m\left[N_1(y_1) - N_1(z_1)\right] + \left[N_2(y_2) - N_2(z_2)\right] \geq 0$$

First, note that $y_1 = z_1$ implies that $N_1(y_1) = N_1(z_1)$. Second, note that $y_2 \geqslant z_2$ implies that $N_2(y_2) \geq N_2(z_2)$. Therefore, the above inequality becomes

$$0 + \underbrace{\left[N_2(y_2) - N_2(z_2)\right]}_{\geq 0} \geq 0$$

which confirms what we needed to show. Hence, ($y_1 = z_1$ and $y_2 \geqslant z_2$) indeed implies $u(y_1, y_2) \geq u(z_1, z_2)$.

3. Let us now show the opposite direction of implication, i.e., $(y_1, y_2) \succsim (z_1, z_2) \Longleftarrow u(y_1, y_2) \geq u(z_1, z_2)$. First, note that if $u(y_1, y_2) \geq u(z_1, z_2)$, then it must be that $mN_1(y_1) + N_2(y_2) \geq mN_1(z_1) + N_2(z_2)$. Rearranging, we obtain

$$m\left[N_1(y_1) - N_1(z_1)\right] + \left[N_2(y_2) - N_2(z_2)\right] \geq 0$$

Then, note that this inequality can be positive for two different reasons: (1) because $N_1(y_1) > N_1(z_1)$, which implies $y_1 > z_1$; or because (2) $N_1(y_1) = N_1(z_1)$ and $N_2(y_2) \geq N_2(z_2)$, which implies $y_1 = z_1$ and $y_2 \geqslant z_2$. And we know that, by definition, these two cases describe the lexicographic preference relation

$$(y_1, y_2) \succsim (z_1, z_2) \text{ if and only if } \begin{cases} y_1 > z_1, \text{ or if} \\ y_1 = z_1 \text{ and } y_2 \geqslant z_2 \end{cases}$$

Hence, $(y_1, y_2) \succsim (z_1, z_2) \Longleftarrow u(y_1, y_2) \geq u(z_1, z_2)$. Since we have shown this implication in both directions, then we have confirmed that this utility function indeed represents the lexicographic preference relation

$$(y_1, y_2) \succsim (z_1, z_2) \Longleftrightarrow u(y_1, y_2) \geq u(z_1, z_2)$$