Bertrand Model of Price Competition
Bertrand Model of Price Competition

• Consider:
  – An industry with two firms, 1 and 2, selling a homogeneous product
  – Firms face market demand $x(p)$, where $x(p)$ is continuous and strictly decreasing in $p$
  – There exists a high enough price (choke price) $\bar{p} < \infty$ such that $x(p) = 0$ for all $p > \bar{p}$
  – Both firms are symmetric in their constant marginal cost $c > 0$, where $x(c) \in (0, \infty)$
  – Every firm $j$ simultaneously sets a price $p_j$
Bertrand Model of Price Competition

• Firm $j$’s demand is

\[
x_j(p_j, p_k) = \begin{cases} 
  x(p_j) & \text{if } p_j < p_k \\
  \frac{1}{2} x(p_j) & \text{if } p_j = p_k \\
  0 & \text{if } p_j > p_k 
\end{cases}
\]

• *Intuition*: Firm $j$ captures

  – all market if its price is the lowest, $p_j < p_k$
  – no market if its price is the highest, $p_j > p_k$
  – shares the market with firm $k$ if the price of both firms coincide, $p_j = p_k$
Bertrand Model of Price Competition

- Given prices $p_j$ and $p_k$, firm $j$’s profits are therefore
  \[(p_j - c) \cdot x_j(p_j, p_k)\]

- We are now ready to find equilibrium prices in the Bertrand duopoly model.
  - There is a unique NE $(p_j^*, p_k^*)$ in the Bertrand duopoly model. In this equilibrium, both firms set prices equal to marginal cost, $p_j^* = p_k^* = c$. 

Advanced Microeconomic Theory
Bertrand Model of Price Competition

• Let’s us describe the best response function of firm \( j \).
• If \( p_k < c \), firm \( j \) sets its price at \( p_j = c \).
  – Firm \( j \) does not undercut firm \( k \) since that would entail negative profits.
• If \( c < p_k < p_j \), firm \( j \) slightly undercuts firm \( k \), i.e., \( p_j = p_k - \varepsilon \).
  – This allows firm \( j \) to capture all sales and still make a positive margin on each unit.
• If \( p_k > p_m \), where \( p_m \) is a monopoly price, firm \( j \) does not need to charge more than \( p_m \), i.e., \( p_j = p_m \).
  – \( p_m \) allows firm \( j \) to capture all sales and maximize profits as the only firm selling a positive output.
Bertrand Model of Price Competition

- Firm j’s best response has:
  - a flat segment for all \( p_k < c \), where \( p_j(p_k) = c \)
  - a positive slope for all \( c < p_k < p_j \), where firm j charges a price slightly below firm k
  - a flat segment for all \( p_k > p_m \), where \( p_j(p_k) = p_m \)
Bertrand Model of Price Competition

- A symmetric argument applies to the construction of the best response function of firm $k$.
- A mutual best response for both firms is 
  \[(p_1^*, p_2^*) = (c, c)\]
  where the two best response functions cross each other.
- This is the NE of the Bertrand model
  – Firms make no economic profits.
Bertrand Model of Price Competition

• With only two firms competing in prices we obtain the perfectly competitive outcome, where firms set prices equal to marginal cost.

• Price competition makes each firm $j$ face an infinitely elastic demand curve at its rival’s price, $p_k$.
  – Any increase (decrease) from $p_k$ infinitely reduces (increases, respectively) firm $j$’s demand.
Bertrand Model of Price Competition

• How much does Bertrand equilibrium hinge into our assumptions?
  – Quite a lot

• The competitive pressure in the Bertrand model with homogenous products is ameliorated if we instead consider:
  – Price competition (but allowing for heterogeneous products)
  – Quantity competition (still with homogenous products)
  – Capacity constraints
Bertrand Model of Price Competition

• **Remark:**
  
  – How our results would be affected if firms face different production costs, i.e., \( 0 < c_1 < c_2 \)?
  
  – The most efficient firm sets a price equal to the marginal cost of the least efficient firm, \( p_1 = c_2 \).
  
  – Other firms will set a random price in the uniform interval
    \[ [c_1, c_1 + \eta] \]
  where \( \eta > 0 \) is some small random increment with probability distribution \( f(p, \eta) > 0 \) for all \( p \).
Cournot Model of Quantity Competition
Cournot Model of Quantity Competition

• Let us now consider that firms compete in quantities.
• Assume that:
  – Firms bring their output $q_1$ and $q_2$ to a market, the market clears, and the price is determined from the inverse demand function $p(q)$, where $q = q_1 + q_2$.
  – $p(q)$ satisfies $p'(q) < 0$ at all output levels $q \geq 0$,
  – Both firms face a common marginal cost $c > 0$
  – $p(0) > c$ in order to guarantee that the inverse demand curve crosses the constant marginal cost curve at an interior point.
Cournot Model of Quantity Competition

• Let us first identify every firm’s best response function
• Firm 1’s PMP, for a given output level of its rival, \( \bar{q}_2 \),

\[
\max_{q_1 \geq 0} \frac{p(q_1 + \bar{q}_2)q_1 - cq_1}{\text{Price}}
\]

• When solving this PMP, firm 1 treats firm 2’s production, \( \bar{q}_2 \), as a parameter, since firm 1 cannot vary its level.
Cournot Model of Quantity Competition

• FOCs:
  
  \[ p'(q_1 + \bar{q}_2)q_1 + p(q_1 + \bar{q}_2) - c \leq 0 \]
  
  with equality if \( q_1 > 0 \)

• Solving this expression for \( q_1 \), we obtain firm 1’s best response function (BRF), \( q_1(\bar{q}_2) \).

• A similar argument applies to firm 2’s PMP and its best response function \( q_2(\bar{q}_1) \).

• Therefore, a pair of output levels \( (q_1^*, q_2^*) \) is NE of the Cournot model if and only if
  
  \[ q_1^* \in q_1(\bar{q}_2) \] for firm 1’s output
  
  \[ q_2^* \in q_2(\bar{q}_1) \] for firm 2’s output
Cournot Model of Quantity Competition

• To show that \( q_1^*, q_2^* > 0 \), let us work by contradiction, assuming \( q_1^* = 0 \).
  – Firm 2 becomes a monopolist since it is the only firm producing a positive output.

• Using the FOC for firm 1, we obtain
  \[
p'(0 + q_2^*)0 + p(0 + q_2^*) \leq c
  \]
  or \( p(q_2^*) \leq c \)

• And using the FOC for firm 2, we have
  \[
p'(q_2^* + 0)q_2^* + p(q_2^* + 0) \leq c
  \]
  or \( p'(q_2^*)q_2^* + p(q_2^*) \leq c \)

• This implies firm 2’s MR under monopoly is lower than its MC. Thus, \( q_2^* = 0 \).
Cournot Model of Quantity Competition

• Hence, if \( q_1^* = 0 \), firm 2’s output would also be zero, \( q_2^* = 0 \).

• But this implies that \( p(0) < c \) since no firm produces a positive output, thus violating our initial assumption \( p(0) > c \).
  – Contradiction!

• As a result, we must have that both \( q_1^* > 0 \) and \( q_2^* > 0 \).

• *Note:* This result does not necessarily hold when both firms are asymmetric in their production cost.
Cournot Model of Quantity Competition

• **Example** (symmetric costs):
  
  – Consider an inverse demand curve \( p(q) = a - bq \), and two firms competing à la Cournot both facing a marginal cost \( c > 0 \).

  – Firm 1’s PMP is
    \[
    [a - b(q_1 + \bar{q}_2)]q_1 - cq_1
    \]

  – FOC wrt \( q_1 \):
    \[
    a - 2bq_1 - \bar{q}_2 - c \leq 0
    \]
    with equality if \( q_1 > 0 \)
Cournot Model of Quantity Competition

• **Example** (continue):
  
  – Solving for \( q_1 \), we obtain firm 1’s BRF
    
    \[
    q_1(\bar{q}_2) = \frac{a-c}{2b} - \frac{\bar{q}_2}{2}
    \]
  
  – Analogously, firm 2’s BRF
    
    \[
    q_2(\bar{q}_1) = \frac{a-c}{2b} - \frac{\bar{q}_1}{2}
    \]
Cournot Model of Quantity Competition

- Firm 1’s BRF:
  - When $q_2 = 0$, then $q_1 = \frac{a-c}{2b}$, which coincides with its output under monopoly.
  - As $q_2$ increases, $q_1$ decreases (i.e., firm 1’s and 2’s output are strategic substitutes)
  - When $q_2 = \frac{a-c}{b}$, then $q_1 = 0$. 
A similar argument applies for firm 2’s BRF.

Superimposing both firms’ BRFs, we obtain the Cournot equilibrium output pair \((q_1^*, q_2^*)\).
Cournot Model of Quantity Competition

\[ q_1 + q_2 = q_c = \frac{a - c}{b} \]

Perfect competition

\[ q_1^* + q_2^* = q_c = \frac{a - c}{b} \]

Monopoly

\[ q_1^* + q_2^* = q_m = \frac{a - c}{2b} \]
Cournot Model of Quantity Competition

- Cournot equilibrium output pair \((q_1^*, q_2^*)\) occurs at the intersection of the two BRFs, i.e.,

\[
(q_1^*, q_2^*) = \left( \frac{a-c}{3b}, \frac{a-c}{3b} \right)
\]

- Aggregate output becomes

\[
q^* = q_1^* + q_2^* = \frac{a-c}{3b} + \frac{a-c}{3b} = \frac{2(a-c)}{3b}
\]

which is larger than under monopoly, \(q_m = \frac{a-c}{2b}\), but smaller than under perfect competition, \(q_c = \frac{a-c}{b}\).
Cournot Model of Quantity Competition

• The equilibrium price becomes

\[ p(q^*) = a - bq^* = a - b \left( \frac{2(a-c)}{3b} \right) = \frac{a+2c}{3} \]

which is lower than under monopoly, \( p_m = \frac{a+c}{2} \), but higher than under perfect competition, \( p_c = c \).

• Finally, the equilibrium profits of every firm \( j \)

\[ \pi_j^* = p(q^*)q_j^* - cq_j^* = \left( \frac{a+2c}{3} \right) \left( \frac{a-c}{3b} \right) - c \left( \frac{a-c}{3b} \right) = \frac{(a-c)^2}{4b} \]

which are lower than under monopoly, \( \pi_m = \frac{(a-c)^2}{4b} \), but higher than under perfect competition, \( \pi_c = 0 \).
Cournot Model of Quantity Competition

- Quantity competition (Cournot model) yields less competitive outcomes than price competition (Bertrand model), whereby firms’ behavior mimics that in perfectly competitive markets.
  - That’s because, the demand that every firm faces in the Cournot game is not infinitely elastic.
  - A reduction in output does not produce an infinite increase in market price, but instead an increase of $- p' (q_1 + q_2)$.
  - Hence, if firms produce the same output as under marginal cost pricing, i.e., half of $\frac{a-c}{2}$, each firm would have incentives to deviate from such a high output level by marginally reducing its output.
Cournot Model of Quantity Competition

• Equilibrium output under Cournot does not coincide with the monopoly output either.
  – That’s because, every firm $i$, individually increasing its output level $q_i$, takes into account how the reduction in market price affects its own profits, but ignores the profit loss (i.e., a negative external effect) that its rival suffers from such a lower price.
  – Since every firm does not take into account this external effect, aggregate output is too large, relative to the output that would maximize firms’ joint profits.
Cournot Model of Quantity Competition

• **Example** (Cournot vs. Cartel):
  
  – Let us demonstrate that firms’ Cournot output is larger than that under the cartel.
  
  – PMP of the cartel is
    \[
    \max_{q_1, q_2} \left[ (a - b(q_1 + q_2))q_1 - cq_1 \right] \\
    + \left[ (a - b(q_1 + q_2))q_2 - cq_2 \right]
    \]
  
  – Since \( Q = q_1 + q_2 \), the PMP can be written as
    \[
    \max_{q_1, q_2} (a - b(q_1 + q_2))(q_1 + q_2) - c(q_1 + q_2)
    = \max_Q (a - bQ)Q - cQ = aQ - bQ^2 - cQ
    \]
Cournot Model of Quantity Competition

• Example (continued):
  – FOC wrt $Q$
    
    $a - 2bQ - c \leq 0$
  – Solving for $Q$, we obtain the aggregate output
    
    $Q^* = \frac{a-c}{2b}$
    
    which is positive since $a > c$, i.e., $p(0) = a > c$.
  – Since firms are symmetric in costs, each produces
    
    $q_i = \frac{Q}{2} = \frac{a-c}{4b}$
Cournot Model of Quantity Competition

• **Example** (continued):
  – The equilibrium price is
    \[ p = a - bQ = a - b \frac{a-c}{2b} = \frac{a+c}{2} \]
  – Finally, the equilibrium profits are
    \[ \pi_i = p \cdot q_i - cq_i \]
    \[ = \frac{a+c}{2} \cdot \frac{a-c}{4b} - c \frac{a-c}{4b} = \frac{(a-c)^2}{8b} \]
    which is larger than firms would obtain under Cournot competition, \( \frac{(a-c)^2}{9b} \).
Cournot Model of Quantity Competition: Cournot Pricing Rule

• Firms’ market power can be expressed using a variation of the Lerner index.
  – Consider firm $j$’s profit maximization problem
    $$\pi_j = p(q)q_j - c_j(q_j)$$
  – FOC for every firm $j$
    $$p'(q)q_j + p(q) - c_j = 0$$
    or $$p(q) - c_j = -p'(q)q_j$$
  – Multiplying both sides by $q$ and dividing them by $p(q)$ yield
    $$\frac{q}{p(q)} \frac{p(q) - c_j}{p(q)} = \frac{-p'(q)q_j}{p(q)} q$$
Cournot Model of Quantity Competition: Cournot Pricing Rule

– Recalling $\frac{1}{\varepsilon} = -p'(q) \cdot \frac{q}{p(q)}$, we have

$$q \frac{p(q) - c_j}{p(q)} = \frac{1}{\varepsilon} q_j$$

or

$$\frac{p(q) - c_j}{p(q)} = \frac{1}{\varepsilon} \frac{q_j}{q}$$

– Defining $\alpha_j \equiv \frac{q_j}{q}$ as firm $j$’s market share, we obtain

$$\frac{p(q) - c_j}{p(q)} = \frac{\alpha_j}{\varepsilon}$$

which is referred to as the Cournot pricing rule.
Cournot Model of Quantity Competition: Cournot Pricing Rule

– Note:

- When $\alpha_j = 1$, implying that firm $j$ is a monopoly, the IEPR becomes a special case of the Cournot price rule.
- The larger the market share $\alpha_j$ of a given firm, the larger the price markup of firm $j$.
- The more inelastic demand $\varepsilon$ is, the larger the price markup of firm $j$. 
Cournot Model of Quantity Competition: Cournot Pricing Rule

**Example** (Merger effects on Cournot Prices):

- Consider an industry with $n$ firms and a constant-elasticity demand function $q(p) = ap^{-1}$, where $a > 0$ and $\varepsilon = 1$.

- Before merger, we have
  
  \[
  \frac{p^B - c}{p^B} = \frac{1}{n} \implies p^B = \frac{nc}{n-1}
  \]

- After the merger of $k < n$ firms $n - k + 1$ firms remain in the industry, and thus
  
  \[
  \frac{p^A - c}{p^A} = \frac{1}{n - k + 1} \implies p^A = \frac{(n - k + 1)c}{n - k}
  \]
Cournot Model of Quantity Competition: Cournot Pricing Rule

• **Example** (continued):
  
  – The percentage change in prices is

  \[
  \%\Delta p = \frac{p^A - p^B}{p^B} = \frac{(n - k + 1)c}{n - k} - \frac{nc}{n - 1}
  \]

  \[
  = \frac{k - 1}{n(n - k)} > 0
  \]

  – Hence, prices increase after the merger.

  – Also, \%\Delta p increases as the number of merging firms \( k \) increases

  \[
  \frac{\partial \%\Delta p}{\partial k} = \frac{n - 1}{n(n - k)^2} > 0
  \]
Cournot Model of Quantity Competition: Cournot Pricing Rule

• **Example** (continued):
  – The percentage increase in price after the merger, \(\%\Delta p\), as a function of the number of merging firms, \(k\).
  – For simplicity, \(n = 100\).
Cournot Model of Quantity Competition: Asymmetric Costs

• Assume that firm 1 and 2’s constant marginal costs of production differ, i.e., $c_1 > c_2$, so firm 2 is more efficient than firm 1. Assume also that the inverse demand function is $p(Q) = a - bQ$, and $Q = q_1 + q_2$.

• Firm $i$’s PMP is
  \[
  \max_{q_i} (a - b(q_i + q_j))q_i - c_i q_i
  \]

• FOC:
  \[
  a - 2bq_i - bq_j - c_i = 0
  \]
Cournot Model of Quantity Competition: Asymmetric Costs

- Solving for $q_i$ (assuming an interior solution) yields firm $i$’s BRF
  \[ q_i(q_j) = \frac{a - c_i}{2b} - \frac{q_j}{2} \]

- Firm 1’s optimal output level can be found by plugging firm 2’s BRF into firm 1’s
  \[ q_1^* = \frac{a - c_1}{2b} - \frac{1}{2} \left( \frac{a - c_2}{2b} - \frac{q_1^*}{2} \right) \iff q_1^* = \frac{a - 2c_1 + c_2}{3b} \]

- Similarly, firm 2’s optimal output level is
  \[ q_2^* = \frac{a - c_2}{2b} - \frac{q_1^*}{2} = \frac{a + c_1 - 2c_2}{3b} \]
Cournot Model of Quantity Competition: Asymmetric Costs

• The output levels \((q_1^*, q_2^*)\) also vary when \((c_1, c_2)\) changes

\[
\frac{\partial q_1^*}{\partial c_1} = -\frac{2}{3b} < 0 \quad \text{and} \quad \frac{\partial q_1^*}{\partial c_2} = \frac{1}{3b} > 0
\]

\[
\frac{\partial q_2^*}{\partial c_1} = \frac{1}{3b} > 0 \quad \text{and} \quad \frac{\partial q_2^*}{\partial c_2} = -\frac{2}{3b} < 0
\]

• *Intuition*: Each firm’s output decreases in its own costs, but increases in its rival’s costs.
Cournot Model of Quantity Competition: Asymmetric Costs

- BRFs for firms 1 and 2 when \( c_1 > \frac{a+c_2}{2} \) (i.e., only firm 2 produces).
- BRFs cross at the vertical axis where \( q_1^* = 0 \) and \( q_2^* > 0 \) (i.e., a corner solution)
Cournot Model of Quantity Competition: $J > 2$ firms

- Consider $J > 2$ firms, all facing the same constant marginal cost $c > 0$. The linear inverse demand curve is $p(Q) = a - bQ$, where $Q = \sum_j q_k$.
- Firm $i$’s PMP is
  \[
  \max_{q_i} \left[ a - b \left( q_i + \sum_{k \neq i} q_k \right) \right] q_i - c q_i
  \]
- FOC:
  \[
  a - 2bq_i^* - b \sum_{k \neq i} q_k^* - c \leq 0
  \]
Cournot Model of Quantity Competition: 
$J > 2$ firms

• Solving for $q_i^*$, we obtain firm $i$’s BRF

$$q_i^* = \frac{a - c}{2b} - \frac{1}{2} \sum_{k \neq i} q_k^*$$

• Since all firms are symmetric, their BRFs are also symmetric, implying $q_1^* = q_2^* = \cdots = q_J^*$. This implies $\sum_{k \neq i} q_k^* = Jq_i^* - q_i^* = (J - 1)q_i^*$.

• Hence, the BRF becomes

$$q_i^* = \frac{a - c}{2b} - \frac{1}{2} (J - 1)q_i^*$$
Cournot Model of Quantity Competition: 
\( J > 2 \) firms

• Solving for \( q_i^* \)

\[
q_i^* = \frac{a - c}{(J + 1)b}
\]

which is also the equilibrium output for other \( J - 1 \) firms.

• Therefore, aggregate output is

\[
Q^* = J q_i^* = \frac{J}{J + 1} \frac{a - c}{b}
\]

and the corresponding equilibrium price is

\[
p^* = a - bQ^* = \frac{a + Jc}{J + 1}
\]
Cournot Model of Quantity Competition: $J > 2$ firms

- Firm $i$’s equilibrium profits are

$$\pi_i^* = (a - bQ^*)q_i^* - cq_i^*$$

$$= \left( a - b \left( \frac{J}{J + 1} \frac{a - c}{b} \right) \right) \left( \frac{a - c}{(J + 1)b} \right) - c \left( \frac{a - c}{(J + 1)b} \right)$$

$$= \left( \frac{a - c}{(J + 1)b} \right)^2 = (q_i^*)^2$$
Cournot Model of Quantity Competition: $J > 2$ firms

• We can show that

$$\lim_{J \to 2} q_i^* = \frac{a - c}{(2 + 1)b} = \frac{a - c}{3b}$$

$$\lim_{J \to 2} Q^* = \frac{2(a - c)}{(2 + 1)b} = \frac{2(a - c)}{3b}$$

$$\lim_{J \to 2} p^* = \frac{a + 2c}{(2 + 1)} = \frac{a + 2c}{3}$$

which exactly coincide with our results in the Cournot duopoly model.
Cournot Model of Quantity Competition: \( J > 2 \) firms

• We can show that

\[
\lim_{J \to 1} q_i^* = \frac{a - c}{(1 + 1)b} = \frac{a - c}{2b}
\]

\[
\lim_{J \to 1} Q^* = \frac{1(a - c)}{(1 + 1)b} = \frac{a - c}{2b}
\]

\[
\lim_{J \to 1} p^* = \frac{a + 1c}{(1 + 1)} = \frac{a + c}{2}
\]

which exactly coincide with our findings in the monopoly.
Cournot Model of Quantity Competition: $J > 2$ firms

- We can show that
  \[ \lim_{J \to \infty} q_i^* = 0 \]
  \[ \lim_{J \to \infty} Q^* = \frac{a - c}{b} \]
  \[ \lim_{J \to \infty} p^* = c \]

  which coincides with the solution in a perfectly competitive market.
Product Differentiation
Product Differentiation

• So far we assumed that firms sell homogenous (undifferentiated) products.

• What if the goods firms sell are differentiated?
  – For simplicity, we will assume that product attributes are exogenous (not chosen by the firm)
Product Differentiation: Bertrand Model

• Consider the case where every firm $i$, for $i = \{1,2\}$, faces demand curve

\[ q_i(p_i, p_j) = a - bp_i + cp_j \]

where $a, b, c > 0$ and $j \neq i$.

• Hence, an increase in $p_j$ increases firm $i$’s sales.

• Firm $i$’s PMP:

\[ \max_{p_i \geq 0} (a - bp_i + cp_j)p_i \]

• FOC:

\[ -2bp_i + cp_j = 0 \]
Product Differentiation: Bertrand Model

• Solving for $p_i$, we find firm $i$’s BRF

$$p_i(p_j) = \frac{a + cp_j}{2b}$$

• Firm $j$ also has a symmetric BRF.

• Note:
  – BRFs are now positively sloped
  – An increase in firm $j$’s price leads firm $i$ to increase his, and vice versa
  – In this case, firms’ choices (i.e., prices) are strategic complements
Product Differentiation: Bertrand Model

\[ p_2(p_1) \]

\[ p_1(p_2) \]
Product Differentiation: Bertrand Model

• Simultaneously solving the two BRS yields

\[ p_i^* = \frac{a}{2b - c} \]

with corresponding equilibrium sales of

\[ q_i^*(p_i^*, p_j^*) = a - bp_i^* + cp_j^* = \frac{ab}{2b - c} \]

and equilibrium profits of

\[ \pi_i^* = p_i^* \cdot q_i^*(p_i^*, p_j^*) = \left( \frac{a}{2b - c} \right) \left( \frac{ab}{2b - c} \right) = \frac{a^2 b}{(2b - c)^2} \]
Product Differentiation: Cournot Model

• Consider two firms with the following linear inverse demand curves

\[ p_1(q_1, q_2) = \alpha - \beta q_1 - \gamma q_2 \] for firm 1
\[ p_2(q_1, q_2) = \alpha - \gamma q_1 - \beta q_2 \] for firm 2

• We assume that \( \beta > 0 \) and \( \beta > \gamma \)
  – That is, the effect of increasing \( q_1 \) on \( p_1 \) is larger than the effect of increasing \( q_1 \) on \( p_2 \)
  – Intuitively, the price of a particular brand is more sensitive to changes in its own output than to changes in its rival’s output
  – In other words, \textit{own-price effects} dominate the \textit{cross-price effects}.  

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Product Differentiation: Cournot Model

- Firm $i$’s PMP is (assuming no costs)
  \[
  \max_{q_i \geq 0} \ (\alpha - \beta q_i - \gamma q_j) q_i
  \]

- FOC:
  \[
  \alpha - 2\beta q_i - \gamma q_j = 0
  \]

- Solving for $q_i$ we find firm $i$’s BRF
  \[
  q_i(q_j) = \frac{\alpha}{2\beta} \ - \ \frac{\gamma}{2\beta} q_j
  \]

- Firm $j$ also has a symmetric BRF
Product Differentiation:
Cournot Model

\[
q_1(q_2) \quad q_2(q_1)
\]

\[
\begin{align*}
q_1^* & = \frac{\alpha}{2\beta} \\
q_2^* & = \frac{\alpha}{\gamma}
\end{align*}
\]
Product Differentiation: Cournot Model

• Comparative statics of firm \( i \)'s BRF
  – As \( \beta \rightarrow \gamma \) (products become more homogeneous), BRF becomes steeper. That is, the profit-maximizing choice of \( q_i \) is more sensitive to changes in \( q_j \) (tougher competition)
  – As \( \gamma \rightarrow 0 \) (products become very differentiated), firm \( i \)'s BRF no longer depends on \( q_j \) and becomes flat (milder competition)
Product Differentiation: Cournot Model

- Simultaneously solving the two BRF yields
  \[ q_i^* = \frac{\alpha}{2\beta + \gamma} \text{ for all } i = \{1,2\} \]
  with a corresponding equilibrium price of
  \[ p_i^* = \alpha - \beta q_i^* - \gamma q_j^* = \frac{\alpha \beta}{2\beta + \gamma} \]
  and equilibrium profits of
  \[ \pi_i^* = p_i^* q_i^* = \left( \frac{\alpha \beta}{2\beta + \gamma} \right) \left( \frac{\alpha}{2\beta + \gamma} \right) = \frac{\alpha^2 \beta}{(2\beta + \gamma)^2} \]
Product Differentiation: Cournot Model

• **Note:**
  – As $\gamma$ increases (products become more homogeneous), individual and aggregate output decrease, and individual profits decrease as well.
  – If $\gamma \to \beta$ (indicating undifferentiated products), then $q_i^* = \frac{\alpha}{2\beta + \beta} = \frac{\alpha}{3\beta}$ as in standard Cournot models of homogeneous products.
  – If $\gamma \to 0$ (extremely differentiated products), then $q_i^* = \frac{\alpha}{2\beta + 0} = \frac{\alpha}{2\beta}$ as in monopoly.
Dynamic Competition
Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

• Assume that firm 1 chooses its price $p_1$ first, whereas firm 2 observes that price and responds with its own price $p_2$.

• Since the game is a sequential-move game (rather than a simultaneous-move game), we should use *backward induction*. 
Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

• Firm 2 (the follower) has a BRF given by

\[ p_2(p_1) = \begin{cases} 
  p_1 - \varepsilon & \text{if } p_1 > c \\
  c & \text{if } p_1 \leq c 
\end{cases} \]

while firm 1’s (the leader’s) BRF is

\[ p_1 = c \]

• Intuition: the follower undercuts the leader’s price \( p_1 \) by a small \( \varepsilon > 0 \) if \( p_1 > c \), or keeps it at \( p_2 = c \) if the leader sets \( p_1 = c \).
Dynamic Competition: Sequential Bertrand Model with Homogeneous Products

• The leader expects that its price will be:
  – undercut by the follower when $p_1 > c$ (thus yielding no sales)
  – mimicked by the follower when $p_1 = c$ (thus entailing half of the market share)

• Hence, the leader has (weak) incentives to set a price $p_1 = c$.

• As a consequence, the equilibrium price pair remains at $(p_1^*, p_2^*) = (c, c)$, as in the simultaneous-move version of the Bertrand model.
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• Assume that firms sell differentiated products, where firm $j$’s demand is
  
  $$q_j = D_j(p_j, p_k)$$

  – Example: $q_j(p_j, p_k) = a - bp_j + cp_k$, where $a, b, c > 0$ and $b > c$

• In the second stage, firm 2 (the follower) solves following PMP

  $$\max_{p_2 \geq 0} \pi_2 = p_2 q_2 - TC(q_2)$$

  $$= p_2 D_2(p_2, p_1) - TC(D_2(p_2, p_1))$$
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• FOCs wrt $p_2$ yield

$$D_2(p_2, p_1) + p_2 \frac{\partial D_2(p_2, p_1)}{\partial p_2}$$

$$\left[ - \frac{\partial TC(D_2(p_2, p_1))}{\partial D_2(p_2, p_1)} \frac{\partial D_2(p_2, p_1)}{\partial p_2} \right]$$

Using the chain rule

$$= 0$$

• Solving for $p_2$ produces the follower’s BRF for every price set by the leader, $p_1$, i.e., $p_2(p_1)$. 

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Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• In the first stage, firm 1 (leader) anticipates that the follower will use BRF $p_2(p_1)$ to respond to each possible price $p_1$, hence solves following PMP

$$\max_{p_1 \geq 0} \pi_1 = p_1 q_1 - TC(q_1)$$

$$= p_1 D_1 \left( p_1, \underbrace{p_2(p_1)}_{BRF_2} \right) - TC \left( \frac{q_1}{D_1(p_1, p_2(p_1))} \right)$$
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• FOCs wrt $p_1$ yield

$$D_1(p_1, p_2) + p_1 \left[ \frac{\partial D_1(p_1, p_2)}{\partial p_1} + \frac{\partial D_1(p_1, p_2)}{\partial p_2(p_1)} \frac{\partial p_2(p_1)}{\partial p_1} \right] - \frac{\partial TC(D_1(p_1, p_2))}{\partial D_1(p_1, p_2)} \left[ \frac{\partial D_1(p_1, p_2)}{\partial p_1} + \frac{\partial D_1(p_1, p_2)}{\partial p_2(p_1)} \frac{\partial p_2(p_1)}{\partial p_1} \right] = 0$$

• Or more compactly as

$$D_1(p_1, p_2) + \left( p_1 - \frac{\partial TC(D_1(p_1, p_2))}{\partial D_1(p_1, p_2)} \right) \frac{\partial D_1(p_1, p_2)}{\partial p_1} \left[ 1 + \frac{\partial p_2(p_1)}{\partial p_1} \right] = 0$$
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• In contrast to the Bertrand model with simultaneous price competition, an increase in firm 1’s price now produces an increase in firm 2’s price in the second stage.

• Hence, the leader has more incentives to raise its price, ultimately softening the price competition.

• While a softened competition benefits both the leader and the follower, the real beneficiary is the follower, as its profits increase more than the leader’s.
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• **Example:**
  
  – Consider a linear demand \( q_i = 1 - 2p_i + p_j \), with no marginal costs, i.e., \( c = 0 \).
  
  – *Simultaneous Bertrand model*: the PMP is
    \[
    \max_{p_j \geq 0} \pi_j = p_j \cdot (1 - 2p_j + p_k) \text{ for any } k \neq j
    \]
    where FOC wrt \( p_j \) produces firm \( j \)'s BRF
    \[
    p_j(p_k) = \frac{1}{4} + \frac{1}{4} p_k
    \]
    
  – Simultaneously solving the two BRFs yields \( p_j^* = \frac{1}{3} \approx 0.33 \), entailing equilibrium profits of \( \pi_j^* = \frac{2}{9} \approx 0.222 \).
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• **Example** (continued):
  
  – *Sequential Bertrand model*: in the second stage, firm 2’s (the follower’s) PMP is

  \[
  \max_{p_2 \geq 0} \pi_2 = p_2 \cdot (1 - 2p_2 - p_1)
  \]

  where FOC wrt \( p_2 \) produces firm 2’s BRF

  \[
  p_2(p_1) = \frac{1}{4} + \frac{1}{4}p_1
  \]

  – In the first stage, firm 1’s (the leader’s) PMP is

  \[
  \max_{p_1 \geq 0} \pi_1 = p_1 \cdot \left[1 - 2p_1 + \left(\frac{1}{4} + \frac{1}{4}p_1\right)\right] = p_1 \cdot \left[\frac{1}{4} (5 - 7p_1)\right]
  \]
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• **Example** (continued):
  
  – FOC wrt $p_1$, and solving for $p_1$, produces firm 1’s equilibrium price $p_1^* = \frac{5}{14} = 0.36$.
  
  – Substituting $p_1^*$ into the BRF of firm 2 yields $p_2^*(0.36) = \frac{1}{4} + \frac{1}{4}(0.36) = 0.34$.
  
  – Equilibrium profits are hence

\[
\begin{align*}
\pi_1^* &= 0.36 \left[ \frac{1}{4} \left( 5 - 7(0.36) \right) \right] = 0.223 \text{ for firm 1} \\
\pi_2^* &= 0.34 \left( 1 - 2(0.34) + (0.36) \right) = 0.230 \text{ for firm 2}
\end{align*}
\]
Dynamic Competition: Sequential Bertrand Model with Heterogeneous Products

• **Example** (continued):
  – Both firms’ prices and profits are higher in the sequential than in the simultaneous game.
  – However, the follower earns more than the leader in the sequential game! *(second mover’s advantage)*
Dynamic Competition: Sequential Cournot Model with Homogenous Products

- **Stackelberg model**: firm 1 (the leader) chooses output level $q_1$, and firm 2 (the follower) observing the output decision of the leader, responds with its own output $q_2(q_1)$.

- By backward induction, the follower’s BRF is $q_2(q_1)$ for any $q_1$.

- Since the leader anticipates $q_2(q_1)$ from the follower, the leader’s PMP is

$$\max_{q_1 \geq 0} \quad p \left( q_1 + \frac{q_2(q_1)}{\text{BRF}_2} \right) q_1 - TC_1(q_1)$$
Dynamic Competition: Sequential Cournot Model with Homogenous Products

- FOCs wrt $q_1$ yields

\[
p(q_1 + q_2(q_1)) + p'(q_1 + q_2(q_1)) \left[ q_1 + \frac{\partial q_2(q_1)}{\partial q_1} \right] q_1 - \frac{\partial TC_1(q_1)}{\partial q_1} = 0
\]

or more compactly

\[
p(Q) + p'(Q)q_1 + p'(Q) \frac{\partial q_2(q_1)}{\partial q_1} q_1 - \frac{\partial TC_1(q_1)}{\partial q_1} = 0
\]

- This FOC coincides with that for standard Cournot model with simultaneous output decisions, except for the strategic effect.
Dynamic Competition: Sequential Cournot Model with Homogenous Products

• The strategic effect is positive since \( p'(Q) < 0 \) and \( \frac{\partial q_2(q_1)}{\partial q_1} < 0 \).

• Firm 1 (the leader) has more incentive to raise \( q_1 \) relative to the Cournot model with simultaneous output decision.

• **Intuition (first-mover advantage):**
  – By overproducing, the leader forces the follower to reduce its output \( q_2 \) by the amount \( \frac{\partial q_2(q_1)}{\partial q_1} \).
  – This helps the leader sell its production at a higher price, as reflected by \( p'(Q) \); ultimately earning a larger profit than in the standard Cournot model.
Dynamic Competition: Sequential Cournot Model with Homogeneous Products

• **Example:**

  – Consider linear inverse demand $p = a - Q$, where $Q = q_1 + q_2$, and a constant marginal cost of $c$.
  – Firm 2’s (the follower’s) PMP is
    \[
    \max_{q_2} (a - q_1 - q_2)q_2 - cq_2
    \]
  – FOC:
    \[
    a - q_1 - 2q_2 - c = 0
    \]
  – Solving for $q_2$ yields the follower’s BRF
    \[
    q_2(q_1) = \frac{a - q_1 - c}{2}
    \]
Dynamic Competition: Sequential Cournot Model with Homogenous Products

• **Example** (continued):
  – Plugging $q_2(q_1)$ into the leader’s PMP, we get
    $$\max_{q_1} \left( a - q_1 - \frac{a-q_1-c}{2} \right) q_1 - cq_1 = \frac{1}{2} (a - q_1 - c)$$
  – FOC:
    $$\frac{1}{2} (a - 2q_1 - c) = 0$$
  – Solving for $q_1$, we obtain the leader’s equilibrium output level $q_1^* = \frac{a-c}{2}$.
  – Substituting $q_1^*$ into the follower’s BRF yields the follower’s equilibrium output $q_2^* = \frac{a-c}{4}$. 

Advanced Microeconomic Theory
Dynamic Competition: Sequential Cournot Model with Homogenous Products

\[ q_1 = \frac{a - c}{2} \]

\[ q_2(q_1) \]

\[ q_1(q_2) \]
Dynamic Competition: Sequential Cournot Model with Homogenous Products

• **Example** (continued):
  - The equilibrium price is
    \[ p = a - q_1^* - q_2^* = \frac{a + 3c}{4} \]
  - And the resulting equilibrium profits are
    \[ \pi_1^* = \left( \frac{a+3c}{4} \right) \left( \frac{a-c}{2} \right) - c \left( \frac{a-c}{2} \right) = \frac{(a-c)^2}{8} \]
    \[ \pi_2^* = \left( \frac{a+3c}{4} \right) \left( \frac{a-c}{4} \right) - c \left( \frac{a-c}{4} \right) = \frac{(a-c)^2}{16} \]
Dynamic Competition: Sequential Cournot Model with Homogenous Products

- Linear inverse demand $p(Q) = a - Q$
- Symmetric marginal costs $c > 0$

\[ p^m = \frac{a + c}{2} \]
\[ p^{Cournot} = \frac{a + 2c}{3} \]
\[ p^{Stackelberg} = \frac{a + 3c}{4} \]
\[ p^{P.C.} = p^{Bertrand} = c \]

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Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

• Assume that firms sell differentiated products, with inverse demand curves for firms 1 and 2

\[ p_1(q_1, q_2) = \alpha - \beta q_1 - \gamma q_2 \text{ for firm 1} \]
\[ p_2(q_1, q_2) = \alpha - \gamma q_1 - \beta q_2 \text{ for firm 2} \]

• Firm 2’s (the follower’s) PMP is

\[ \max_{q_2} (\alpha - \gamma q_1 - \beta q_2) \cdot q_2 \]

where, for simplicity, we assume no marginal costs.

• FOC:

\[ \alpha - \gamma q_1 - 2\beta q_2 = 0 \]
Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

- Solving for $q_2$ yields firm 2’s BRF
  
  $$q_2(q_1) = \frac{\alpha - \gamma q_1}{2\beta}$$

- Plugging $q_2(q_1)$ into the leader’s firm 1’s (the leader’s) PMP, we get

  $$\max_{q_1} \left( \alpha - \beta q_1 - \gamma \left( \frac{\alpha - \gamma q_1}{2\beta} \right) \right) q_1 =$$

  $$\max_{q_1} \left( \alpha \left( \frac{2\beta - \gamma}{2\beta} \right) - \left( \frac{2\beta^2 - \gamma^2}{2\beta} \right) q_1 \right) q_1$$

- FOC:

  $$\alpha \left( \frac{2\beta - \gamma}{2\beta} \right) - \left( \frac{2\beta^2 - \gamma^2}{\beta} \right) q_1 = 0$$
Dynamic Competition: Sequential Cournot Model with Heterogeneous Products

• Solving for $q_1$, we obtain the leader’s equilibrium output level $q_1^* = \frac{\alpha(2\beta - \gamma)}{2(2\beta^2 - \gamma^2)}$

• Substituting $q_1^*$ into the follower’s BRF yields the follower’s equilibrium output

$$q_2^* = \frac{\alpha - \gamma q_1^*}{2\beta} = \frac{\alpha(4\beta^2 - 2\beta\gamma - \gamma^2)}{4\beta(2\beta^2 - \gamma^2)}$$

• Note:

  – $q_1^* > q_2^*$
  – If $\gamma \to \beta$ (i.e., the products become more homogeneous), $(q_1^*, q_2^*)$ converge to the standard Stackelberg values.
  – If $\gamma \to 0$ (i.e., the products become very differentiated), $(q_1^*, q_2^*)$ converge to the monopoly output $q_m^* = \frac{\alpha}{2\beta}$. 

Advanced Microeconomic Theory
Capacity Constraints
Capacity Constraints

• How come are equilibrium outcomes in the standard Bertrand and Cournot models so different?

• Do firms really compete in prices without facing capacity constraints?
  – Bertrand model assumes a firm can supply infinitely large amount if its price is lower than its rivals.

• Extension of the Bertrand model:
  – **First stage**: firms set capacities, $\bar{q}_1$ and $\bar{q}_2$, with a cost of capacity $c > 0$
  – **Second stage**: firms observe each other’s capacities and compete in prices, simultaneously setting $p_1$ and $p_2$
Capacity Constraints

• What is the role of capacity constraint?
  – When a firm’s price is lower than its capacity, not all consumers can be served.
  – Hence, sales must be rationed through efficient rationing: the customers with the highest willingness to pay get the product first.

• Intuitively, if $p_1 < p_2$ and the quantity demanded at $p_1$ is so large that $Q(p_1) > q_1$, then the first $q_1$ units are served to the customers with the highest willingness to pay (i.e., the upper segment of the demand curve), while some customers are left in the form of residual demand to firm 2.
Capacity Constraints

- At $p_1$ the quantity demanded is $Q(p_1)$, but only $\bar{q}_1$ units can be served.
- Hence, the residual demand is $Q(p_1) - \bar{q}_1$.
- Since firm 2 sets a price of $p_2$, its demand will be $Q(p_2)$.
- Thus, a portion of the residual demand, i.e., $Q(p_2) - \bar{q}_1$, is captured.
Capacity Constraints

• Hence, firm 2’s residual demand can be expressed as

\[
\begin{cases}
Q(p_2) - \bar{q}_1 & \text{if } Q(p_2) - \bar{q}_1 \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

• Should we restrict \(\bar{q}_1\) and \(\bar{q}_2\) somewhat?
  
  – Yes. A firm will never set a huge capacity if such capacity entails negative profits, independently of the decision of its competitor.
Capacity Constraints

• How to express this rather obvious statement with a simple mathematical condition?
  – The maximal revenue of a firm under monopoly is
    \[ \max_q (a - q)q, \]
    which is maximized at
    \[ q = \frac{a}{2}, \]
    yielding profits of
    \[ \frac{a^2}{4}. \]
  – Maximal revenues are larger than costs if
    \[ \frac{a^2}{4} \geq c\bar{q}_j, \]
    or solving for \( \bar{q}_j \),
    \[ \frac{a^2}{4c} \geq \bar{q}_j. \]
  – Intuitively, the capacity cannot be too high, as otherwise the firm would not obtain positive profits regardless of the opponent’s decision.
Capacity Constraints: Second Stage

• By backward induction, we start with the second stage (pricing game), where firms simultaneously choose prices $p_1$ and $p_2$ as a function of the capacity choices $\bar{q}_1$ and $\bar{q}_2$.

• We want to show that in this second stage, both firms set a common price

$$p_1 = p_2 = p^* = a - \bar{q}_1 - \bar{q}_2$$

where demand equals supply, i.e., total capacity,

$$p^* = a - \bar{Q}, \text{ where } \bar{Q} \equiv \bar{q}_1 + \bar{q}_2$$
Capacity Constraints: Second Stage

• In order to prove this result, we start by assuming that firm 1 sets $p_1 = p^*$. We now need to show that firm 2 also sets $p_2 = p^*$, i.e., it does not have incentives to deviate from $p^*$.

• If firm 2 does not deviate, $p_1 = p_2 = p^*$, then it sells up to its capacity $q_2$.

• If firm 2 reduces its price below $p^*$, demand would exceed its capacity $q_2$. As a result, firm 2 would sell the same units as before, $q_2$, but at a lower price.
Capacity Constraints: Second Stage

• If, instead, firm 2 charges a price above $p^*$, then $p_1 = p^* < p_2$ and its revenues become

$$p_2 \hat{Q}(p_2) = \begin{cases} 
    p_2(a - p_2 - \bar{q}_1) & \text{if } a - p_2 - \bar{q}_1 \geq 0 \\
    0 & \text{otherwise}
\end{cases}$$

• **Note:**
  – This is fundamentally different from the standard Bertrand model without capacity constraints, where an increase in price by a firm reduces its sales to zero.
  – When capacity constraints are present, the firm can still capture a residual demand, ultimately raising its revenues after increasing its price.
Capacity Constraints: Second Stage

• We now find the maximum of this revenue function. FOC wrt $p_2$ yields:

$$a - 2p_2 - \bar{q}_1 = 0 \iff p_2 = \frac{a - \bar{q}_1}{2}$$

• The non-deviating price $p^* = a - \bar{q}_1 - \bar{q}_2$ lies above the maximum-revenue price $p_2 = \frac{a - \bar{q}_1}{2}$ when

$$a - \bar{q}_1 - \bar{q}_2 > \frac{a - \bar{q}_1}{2} \iff a > \bar{q}_1 + 2\bar{q}_2$$

• Since $\frac{a^2}{4c} \geq \bar{q}_j$ (capacity constraint), we can obtain

$$\frac{a^2}{4c} + 2 \frac{a^2}{4c} > \bar{q}_1 + 2\bar{q}_2 \iff \frac{3a^2}{4c} > \bar{q}_1 + 2\bar{q}_2$$
Capacity Constraints: Second Stage

- Therefore, \( a > \bar{q}_1 + 2\bar{q}_2 \) holds if \( a > \frac{3a^2}{4c} \) which, solving for \( a \), is equivalent to \( \frac{4c}{3} > a \).
Capacity Constraints: Second Stage

• When $\frac{4c}{3} > a$ holds, capacity constraint $\frac{a^2}{\frac{4c}{3}} \geq \bar{q}_j$ transforms into $\frac{a^2}{3a^2} > \bar{q}_1 + 2\bar{q}_2$, implying $p^* > p_2 = a - \frac{\bar{q}_1}{2}$.

• Thus, firm 2 does not have incentives to increase its price $p_2$ from $p^*$, since that would lower its revenues.
Capacity Constraints: Second Stage

• In short, firm 2 does not have incentives to deviate from the common price
  \[ p^* = a - \bar{q}_1 - \bar{q}_2 \]

• A similar argument applies to firm 1 (by symmetry).

• Hence, we have found an equilibrium in the pricing stage.
Capacity Constraints: First Stage

• In the first stage (capacity setting), firms simultaneously select their capacities \( \bar{q}_1 \) and \( \bar{q}_2 \).

• Inserting stage 2 equilibrium prices, i.e.,
  \[ p_1 = p_2 = p^* = a - \bar{q}_1 - \bar{q}_2, \]
into firm \( j \)'s profit function yields
  \[ \pi_j(\bar{q}_1, \bar{q}_2) = \underbrace{(a - \bar{q}_1 - \bar{q}_2)\bar{q}_i - c\bar{q}_i}_{p^*} \]
• FOC wrt capacity \( \bar{q}_j \) yields firm \( j \)'s BRF
  \[ \bar{q}_j(\bar{q}_k) = \frac{a - c}{2} - \frac{1}{2} \bar{q}_k \]
Capacity Constraints: First Stage

• Solving the two BRFs simultaneously, we obtain a symmetric solution

\[ \bar{q}_j = \bar{q}_k = \frac{a - c}{3} \]

• These are the same equilibrium predictions as those in the standard Cournot model.

• Hence, capacities in this two-stage game coincide with output decisions in the standard Cournot model, while prices are set equal to total capacity.
Endogenous Entry
Endogenous Entry

• So far the number of firms was exogenous
• What if the number of firms operating in a market is endogenously determined?
• That is, how many firms would enter an industry where
  – They know that competition will be a la Cournot
  – They must incur a fixed entry cost $F > 0$.  

Advanced Microeconomic Theory
Endogenous Entry

• Consider inverse demand function $p(q)$, where $q$ denotes aggregate output.

• Every firm $j$ faces the same total cost function, $c(q_j)$, of producing $q_j$ units.

• Hence, the Cournot equilibrium must be symmetric—Every firm produces the same output level $q(n)$, which is a function of the number of entrants.

• Entry profits for firm $j$ are

$$
\pi_j(n) = p \left( \frac{n \cdot q(n)}{Q} \right) q(n) - \frac{c(q(n))}{p(Q)} \text{ Production Costs} - \frac{F}{\omega} \text{ Fixed Entry Cost}
$$
Endogenous Entry

• Three assumptions (valid under most demand and cost functions):
  – individual equilibrium output $q(n)$ is decreasing in $n$;
  – aggregate output $q \equiv n \cdot q(n)$ increases in $n$;
  – equilibrium price $p(n \cdot q(n))$ remains above marginal costs regardless of the number of entrants $n$. 
Endogenous Entry

• **Equilibrium number of firms:**
  
  – The equilibrium occurs when no more firms have incentives to enter or exit the market, i.e., 
  \[ \pi_j(n^e) = 0. \]
  
  – Note that individual profits decrease in \( n \), i.e.,

\[
\pi'(n) = \left[ p(nq(n)) - c'(q(n)) \right] \frac{\partial q(n)}{\partial n} + q(n)p'(nq(n)) \frac{\partial [nq(n)]}{\partial n} < 0
\]
Endogenous Entry

- **Social optimum:**
  - The social planner chooses the number of entrants $n^0$ that maximizes social welfare.

$$\max_n W(n) \equiv \int_0^{nq(n)} p(s)ds - n \cdot c(q(n)) - n \cdot F$$
Endogenous Entry

- $\int_0^{nq(n)} p(s) \, ds = A + B + C + D$
- $n \cdot c(q(n)) = C + D$
- Social welfare is thus $A + B$ minus total entry costs $n \cdot F$

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Endogenous Entry

– FOC wrt $n$ yields

$$p(nq(n)) \left[ n \frac{\partial q(n)}{\partial n} + q(n) \right] - c(q(n)) - nc'(q(n)) \frac{\partial q(n)}{\partial n} - F = 0$$

or, re-arranging,

$$\pi(n) + n[p(nq(n)) - c'(q(n))] \frac{\partial q(n)}{\partial n} = 0$$

– Hence, marginal increase in $n$ entails two opposite effects on social welfare:

  a) the profits of the new entrant increase social welfare (+, appropriability effect)

  b) the entrant reduces the profits of all previous incumbents in the industry as the individual sales of each firm decreases upon entry (-, business stealing effect)
Endogenous Entry

• The “business stealing” effect is represented by:

\[ n[p(nq(n)) - c'(q(n))] \frac{\partial q(n)}{\partial n} < 0 \]

which is negative since \( \frac{\partial q(n)}{\partial n} < 0 \) and

\[ n[p(nq(n)) - c'(q(n))] > 0 \] by definition.

• Therefore, an additional entry induces a reduction in aggregate output by \( n \frac{\partial q(n)}{\partial n} \), which in turn produces a negative effect on social welfare.
Endogenous Entry

- Given the negative sign of the business stealing effect, we can conclude that

\[ W'(n) = \pi(n) + n[p(nq(n)) - c'(q(n))] \frac{\partial q(n)}{\partial n} < \pi(n) \]

and therefore more firms enter in equilibrium than in the social optimum, i.e., \( n^e > n^o \).
Endogenous Entry

\[ W'(n) = \pi(n) + \text{Business Stealing} \]

\[ \pi(n) \]

\[ n^e (\text{Equilibrium}) \]

\[ n^0 (\text{Soc. Optimal}) \]

\[ n, \text{Number of firms} \]
Endogenous Entry

• **Example:**

  – Consider a linear inverse demand $p(Q) = 1 - Q$ and no marginal costs.
  – The equilibrium quantity in a market with $n$ firms that compete a la Cournot is
    \[ q(n) = \frac{1}{n+1} \]
  – Let’s check if the three assumptions from above hold.
Endogenous Entry

• **Example** (continued):
  
  – First, individual output decreases with entry
    \[ \frac{\partial q(n)}{\partial n} = -\frac{1}{(n+1)^2} < 0 \]
  
  – Second, aggregate output \( nq(n) \) increases with entry
    \[ \frac{\partial [nq(n)]}{\partial n} = \frac{1}{(n+1)^2} > 0 \]
  
  – Third, price lies above marginal cost for any number of firms
    \[ p(n) - c = 1 - n \cdot \frac{1}{n+1} = \frac{1}{n+1} > 0 \text{ for all } n \]
Endogenous Entry

• Example (continued):
  – Every firm earns equilibrium profits of
    \[ \pi(n) = \left( \frac{1}{p(n)} \right) \frac{1}{q(n)} - F = \frac{1}{(n+1)^2} - F \]
  – Since equilibrium profits after entry, \( \frac{1}{(n+1)^2} \), is smaller than 1 even if only one firm enters the industry, \( n = 1 \), we assume that entry costs are lower than 1, i.e., \( F < 1 \).
Endogenous Entry

• **Example** (continued):
  
  – Social welfare is

  \[
  W(n) = \int_0^n (1 - s) ds - n \cdot F
  \]

  \[
  = \left[ (s - \frac{s}{2}) \right]_0^n - n \cdot F
  \]

  \[
  = \frac{n(n + 2)}{2} \left( \frac{1}{n + 1} \right)^2 - n \cdot F
  \]
Endogenous Entry

• Example (continued):
  – The number of firms entering the market in equilibrium, $n^e$, is that solving $\pi(n^e) = 0$,
    \[
    \frac{1}{(n^e + 1)^2} - F = 0 \iff n^e = \frac{1}{\sqrt{F}} - 1
    \]
    whereas the number of firms maximizing social welfare, i.e., $n^o$ solving $W'(n^o) = 0$,
    \[
    W'(n^o) = \frac{1}{(n^o + 1)^3} = 0 \iff n^o = \frac{1}{3\sqrt[3]{F}} - 1
    \]
    where $n^e < n^o$ for all admissible values of $F$, i.e., $F \in [0,1]$.  

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Endogenous Entry

- **Example** (continued):

\[ n^e = \frac{1}{F^{1/2}} - 1 \text{ (Equilibrium)} \]

\[ n^o = \frac{1}{F^{1/3}} - 1 \text{ (Soc. Optimal)} \]