Practice exercises - EconS 527 (10/19/2016)

1. [**Independence axiom and convexity**]. Consider an individual with preferences over lotteries that satisfy the independence axiom. Answer the following questions.

   (a) Show that the independence axiom implies convexity, i.e., for three different lotteries L, L' and L'', if $L \succ L'$ and $L \succ L''$, then $L \succ \alpha L' + (1-\alpha) L''$.

   - From $L \succ L'$ we can apply the independence axiom, and obtain

   $$\alpha L + (1-\alpha)L \succ \alpha L' + (1-\alpha)L$$

   where note that we added $(1-\alpha)L$ on both sides of $L \succ L'$. Similarly, from $L \succ L''$ we can apply the independence axiom to obtain

   $$(1-\alpha)L + \alpha L' \succ (1-\alpha)L'' + \alpha L'$$

   where we added $\alpha L'$ on both sides of the strict preference relationship $L \succ L''$. By transitivity (from the two previous expressions), we have

   $$\alpha L + (1-\alpha)L \succ (1-\alpha)L'' + \alpha L'$$

   and rearranging

   $$L \succ \alpha L' + (1-\alpha)L''$$

   Intuitively, convex preference over lotteries means that if a decision maker prefers a lottery L over either two lotteries, $L^{'}$ or $L^{''}$, then he must also prefer lottery L over a convex combination of these two lotteries, $\alpha L^{'} + (1-\alpha)L^{''}$, i.e., the compound lottery of $L^{'}$ and $L^{''}$.

   (b) Discuss why a decision maker whose preferences violate convexity can be offered a sequence of choices that lead him to a sure loss of money

   - If a decision maker's preferences over lotteries violate convexity, then we must have that for three different lotteries L, L' and L'', where $L \succsim L'$ and $L \succsim L''$, we obtain the opposite result than above; that is

   $$\alpha L' + (1-\alpha) L'' \succ L$$

Note that, if the decision maker initially owns the right to participate in lottery L, he will be willing to pay an amount $\$X$ in order to switch to the compound lottery $\alpha L' + (1-\alpha) L''$ given that $\alpha L' + (1-\alpha) L'' \succ L$. Now he owns the compound lottery $\alpha L' + (1-\alpha) L''$, and either lottery L' or lottery L'' are realized. But we know that the decision maker prefers lottery L to either of these lotteries since

$$L \succsim L' \text{ and } L \succsim L''$$

was an initial assumption of this decision maker's preferences over lotteries. Therefore, he would be willing to pay again $\$Y$ in order to obtain lottery L. Hence, the decision maker is exactly as at the starting point of this sequence of deals (lottery L) and has lost $\$X + \$Y$. We can then repeat the process again and again, and make this individual pay $\$X + \$Y$ dollars, keeping him exactly where he started! Essentially, this type of decision maker could be subject to a systematic explanation (the so-called Dutch books), being wiped out of the market place.

2. [**von-Neumann Morgenstern utility function**]. Let G be the set of compound gambles over a finite set of deterministic payoffs $\{a_1, a_2, ...a_n\} \subset \mathbb{R}_+$. A decision maker's preference relation $\succsim$ over compound gambles can be represented by utility function $v : G \to \mathbb{R}$. Let $g \in G$, and let probability p_i be associated to the corresponding payoff a_i. Finally, consider that the decision maker's utility function $v(\cdot)$ is given by

$$v(g) = (1+a_1)^{p_1} (1+a_1)^{p_2} ...(1+a_n)^{p_n} = \prod_{i=1}^{n} (1+a_i)^{p_i}$$

(a) Show that this is *not* a von Neumann-Morgenstern (vNM) utility function.

- Since $v(g)$ is *not* linear in the probabilities, then $v(g)$ cannot be a vNM expected utility function, with general form

$$v(g) = \sum_{i=1}^{N} p_i u(a_i)$$

(b) Show that the decision maker has the same preference relation as an expected utility maximizer with von-Neumann Morgenstern utility function

$$u(g) = \sum_{i=1}^{n} p_i \ln(1+a_i).$$

- Since $\ln(\cdot)$ is a monotonic transformation of $v(\cdot)$, both functions represent the same preference relation. Applying the monotonic transformation $u(g) = \ln[v(g)]$ to the original function $v(g)$, we obtain

$$\ln\left(\prod_{i=1}^{n}(1+a_i)^{p_i}\right) = \sum_{i=1}^{n} p_i \ln(1+a_i)$$

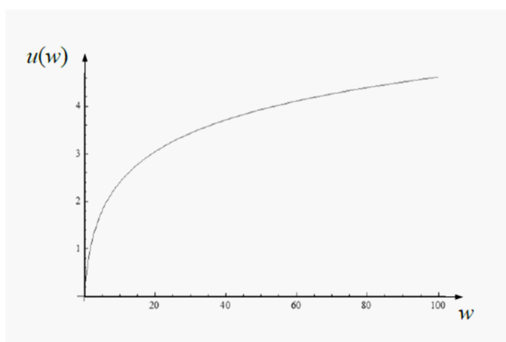
which represents the initial preference relation over lotteries, and it is linear in the probabilities. Hence, it is a vNM utility function.

(c) Assume now that the decision maker you considered in part (b) has utility function $u(w) = \ln(1+w)$ over wealth $w \geqslant 0$. Evaluate his risk attitude (concavity in his utility function). Additionally, find the Arrow-Pratt coefficient of absolute risk aversion, $r_A(w,u)$. How does $r_A(w,u)$ change in wealth?

- Given that $u(w) = \ln(1+w)$, where $w \geqslant 0$, then the first and second order conditions with respect to w are

$$u'(w) = \frac{1}{1+w} > 0 \text{ and } u''(w) = -\frac{1}{(1+w)^2} < 0,$$

which implies that the utility function is concave, as depicted in figure 5.2, and that the decision maker is risk-averse.



Utility function $u(w) = \ln(1+w)$

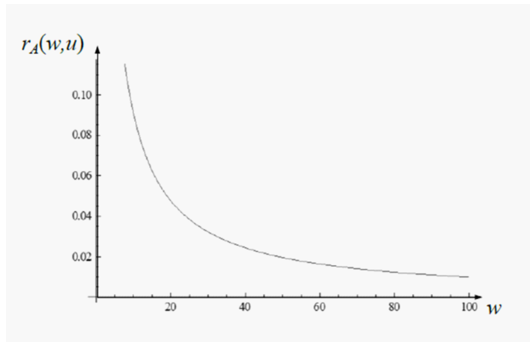
- Let us now obtain the Arrow-Pratt coefficient of absolute risk-aversion, $r_A(w,u)$, as follows

$$r_A(w,u) = -\frac{u''(w)}{u'(w)} = -\frac{-\frac{1}{(1+w)^2}}{\frac{1}{1+w}} = \frac{1}{1+w}$$

- Finally, we want to know how this coefficient of absolute risk aversion varies with wealth,

$$\frac{\partial r_A(w,u)}{\partial w} = -\frac{1}{(1+w)^2}$$

which is negative for all wealth levels $w \geqslant 0$. Hence, the agent becomes less risk-averse as he becomes more wealthy. Figure 5.3 illustrates this coefficient, $r_A(w,u)$, evaluated at different wealth levels.



Coefficient of absolute risk aversion.

3. **[Risk aversion and convexity of indifference curves]** Consider an individual with concave utility function $u(x)$ over monetary outcomes $x \in \mathbb{R}^2_+$. Show that his indifference curves in the (x_1, x_2) −quadrant must be convex.

   - We next first find the expression of indifference curves for a given utility level. Then, we show that indifference curves have a negative slope. Finally, we demonstrate that they are convex (concave, linear) if and only if the utility function $u(x)$ is concave (convex, linear; respectively).
   - Fix the utility level at k, i.e., $u(x) = k$. Solving for x_2, we obtain the indifference curve $x_2 = f(x_1)$. We can express the utility level k as the identity

$$\alpha u(x_1) + (1-\alpha)u(\underbrace{f(x_1)}_{x_2}) \equiv k$$

   - *Slope of the IC.* In order to better understand the shape of this indifference curve, let us differentiate it with respect to x_1, to obtain

$$\alpha u'(x_1) + (1-\alpha)u'(f(x_1))f'(x_1) = 0$$

   and solving for the slope of the indifference curve, $f'(x_1)$, yields

$$f'(x_1) = -\frac{\alpha}{1-\alpha}\frac{u'(x_1)}{u'(f(x_1))}$$

Since $u'(\cdot) > 0$ by definition, and given that $\frac{\alpha}{1-\alpha} > 0$, we obtain that $f'(x_1) < 0$. In words, indifference curves have a negative slope.

- *Convexity of the IC.* Differentiating $f'(x_1)$ again, we find

$$f''(x_1) = -\frac{\alpha}{1-\alpha}\frac{u''(x_1)u'(f(x_1)) - u'(x_1)u''(f(x_1))f'(x_1)}{[u'(f(x_1))]^2}$$

and using the fact that $x_2 = f(x_1)$ and the above result, $f'(x_1) = -\frac{\alpha}{1-\alpha}\frac{u'(x_1)}{u'(f(x_1))}$, this expression becomes

$$f''(x_1) = -\frac{\alpha}{1-\alpha}\frac{u''(x_1)u'(x_2) - u'(x_1)u''(x_2)\left[-\frac{\alpha}{1-\alpha}\frac{u'(x_1)}{u'(x_2)}\right]}{[u'(x_2)]^2}$$

Hence, (1) if the utility function is linear, $u''(x_1) = 0$, the indifference curve is also linear given that $f''(x_1)$ becomes $f''(x_1) = 0$; (2) when the utility function is concave, $u''(x_1) < 0$, the indifference curve is convex because $f''(x_1) > 0$; and (3) when the utility function is convex, $u''(x_1) > 0$, the indifference curve becomes concave as $f''(x_1) < 0$. Therefore, indifference curves are linear, convex or concave when the decision maker is risk neutral, risk averse, and risk lover; respectively.

4. **[Regret theory].** Consider the set of deterministic payoffs $\{a_1, a_2, ...a_n\} \subset \mathbb{R}_+$. Studies in regret-based decision making often consider the following utility function: first, define the highest deterministic payoff that could be reached in gamble g by using function

$$h(g) = \max\{a_k : k \in \{1, 2, ..., n\} \text{ and } p_k > 0\}.$$

Subtracting $h(g)$ from all deterministic outcomes and computing its expected value yields the utility level

$$v(g) = \sum_{i=1}^{n} p_i (a_i - h(g)) = \sum_{i=1}^{n} p_i a_i - h(g)$$

Intuitively, after event i realizes (which provides a payoff a_i to this individual), the "regretful" decision maker compares such monetary payoff with respect to the highest possible payoff he could have obtained from playing this lottery, $h(g)$. Utility functions of this type hence reflect "regret," as individuals experience a disutility from not receiving the highest possible monetary payoff in the lottery.

(a) Compute the expected value of the following two gambles:

$$g^1 = \left(0, 1, 2; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and } g^2 = \left(1, 4, 5; \frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$$

- First, note that $h(g^1) = \max\{0, 1, 2\}$ since all these events can occur with strictly positive probability in lottery g^1. Then, $h(g^1) = 2$, and therefore the individual's expected utility from playing the first gamble, g^1, is

$$v(g^1) = \frac{1}{3}(0-2) + \frac{1}{3}(1-2) + \frac{1}{3}(2-2) = -1$$

  Similarly, we can find the expected utility from playing the second gamble, g^2. In particular, in this case the highest payoff of the lottery is $h(g^2) = \max\{1, 4, 5\} = 5$, implying that the expected utility from this gamble is

$$v(g^2) = \frac{1}{2}(1-5) + \frac{1}{3}(4-5) + \frac{1}{6}(5-5) = -\frac{7}{3}$$

  Note that the individual experiences a lower expected utility from playing the second than the first lottery. Intuitively, this happens because: (1) the distribution of payoffs in the second lottery is more spread than in the first lottery, and this makes the lower payoffs on the second gamble to be compared to a higher possible payoff $h(g^2)$; and (2) because the lowest payoffs on the second gamble are more likely than in the first and, as a consequence, the individual assigns a higher weight in the expected utility calculation to those monetary payoffs in which he is experiencing the biggest regret.

(b) Show that all deterministic outcomes (outcomes with probability 100%) yield the same utility level. That is, $v(a_1) = v(a_2) = ... = v(a_n)$.

- Let us represent by $v(a_i)$ the individual's utility level from a certain deterministic outcome a_i, i.e., $p_{a_i} = 1$. But if outcome a_i occurs with certainty, there is no potential regret. In particular, function $h(g)$ can only find the maximum among all outcomes of the lottery whose probability is strictly greater than zero. Since $p_{a_i} = 1$, then all other outcomes of the lottery receive probability zero, and hence

$$\begin{aligned} h(g) &= \max\{a_k : k \in \{1, 2, ..., n\} \text{ and } p_k > 0\} \\ &= \max\{a_i\} = a_i \end{aligned}$$

Therefore, the individual's expected utility becomes

$$v(a_i) = \sum_{i=1}^{n} p_i \left(a_i - h(g)\right) = 1 \left(a_i - a_i\right) = 0$$

Thus, $v(a_1) = v(a_2) = ... = v(a_n) = 0$, regardless of the monetary payoff associated to outcome a_i. If there is just one event to be regretful about, my expected utility is zero!

(c) Show that the preference relation does not satisfy monotonicity if outcomes are deterministic.

- From the definition of monotonicity, we have that

$$(a_1, a_n; \alpha, 1 - \alpha) \succsim (a_1, a_n; \beta, 1 - \beta)$$

  for all $\alpha, \beta \in [0, 1]$ if and only if $\alpha > \beta$. So if we make $\alpha = 1$ and $\beta = 0$, then the above condition on monotonicity becomes

$$(a_1, a_n; 1, 0) \succsim (a_1, a_n; 0, 1)$$

  Then, clearly $a_1 \succsim a_n$ and $a_1 \not\precsim a_n$, which implies that $a_1 \succ a_n$ strictly.

- However, in part (b) we have shown that the individual's utility is the same (and equal to zero) when outcomes are deterministic. In other words, he is indifferent between gambles whose outcomes are deterministic, i.e., $a_1 \sim a_2 \sim ... \sim a_n$. But this contradicts that $a_1 \succ a_n$ strictly. Therefore, this "regretful" preference relation cannot satisfy monotonicity.